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Alternative vacuum states in static space–times with horizons

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Abstract. The Minkowski, Schwarzschild, and de Sitter geometries possess Killing vectors which are time-like only in certain regions, near whose boundaries (the horizons) the isometries are analogous to Lorentz boosts. In quantum field theory the ground state of the generator of the time-like isometries cannot be 'the physical vacuum' because of its artificial singularities at the horizons. This paper develops several variants of the suggestion of Unruh, to define vacuum initial conditions on the horizon through an analytic property of normal-mode solutions which expresses 'positive frequency' with respect to null translations on the horizon. A theory developed elsewhere of the energy–momentum tensor of the massless scalar field in two-dimensional models is applied to verify that Unruh's condition corresponds to the absence of a flux of energy through an horizon surface, although there may be a flux parallel to the surface. A region with time-like isometries typically is bounded by four such surfaces, two of which may be the usual null infinities, $\mathscr{I}^{\pm}$. In general, Unruh's condition may be applied on two adjacent sides, forcing the appearance of a Hawking flux on the other two. In special cases, however, the opposite horizons can be in 'equilibrium', so that no radiation occurs. In particular, for two-dimensional de Sitter space the vacuum state thus obtained has a stress tensor proportional to the metric times the curvature scalar. If two horizons are not in equilibrium, then no state invariant under the isometries can yield a non-singular stress tensor.

1. Introduction

This paper extends, systematizes, and applies the work of Unruh (1976) (see also Israel 1976, Gibbons and Perry 1977, Damour and Ruffini 1976) on various possible definitions of a physical vacuum state for a quantized scalar field in the Schwarzschild background metric. Near the horizon the geometry of the radius–time plane is essentially flat. In practice this fact is best exploited by adopting Kruskal coordinates, which are related to the usual Schwarzschild coordinates by the same transformation that relates Cartesian coordinates in Minkowski space to Rindler (1966) coordinates (the Fermi coordinates of a uniformly accelerated observer, Manasse and Misner 1963). Unruh concludes that, as in flat space (Fulling 1973), it is naive to identify as the wavefunctions of physical particles those solutions of the field equation which contain only positive frequencies with respect to the time-like Killing vector of the exterior region; instead, the behaviour of solutions on the horizon with respect to Kruskal coordinates must be taken into account. He defines an initial vacuum state through conditions on the behaviour of normal-mode solutions in the far past both on the

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horizon and at infinity; since these solutions do not satisfy the analogous conditions in the far future, the theory predicts a particle-creation phenomenon like that found by Hawking in spherically symmetric gravitational collapse (Hawking 1975). (A similar conclusion was reached by Hartle and Hawking (1976) (see also Hawking 1976) through a different argument.)

The present work begins by generalizing the geometrical situation treated by Unruh; the essential ingredient is a Killing vector which is time-like in some regions and space-like in others, and which acts on the boundary between these regions (a null surface, the horizon) like a homogeneous Lorentz transformation. In particular, space–times are considered in which the spatial infinity is replaced by a second horizon; two-dimensional de Sitter space is the simplest example.

Unruh's treatment of vacuum states is then reviewed and extended. Another possible vacuum is pointed out, corresponding to characterization of 'positive-frequency' solutions entirely by their behaviour on the horizon (cf Israel 1976, Gibbons and Perry 1977). It is argued that this state most closely resembles the ground state of a black hole in equilibrium with its surroundings.

Finally, the expectation value of the energy–momentum tensor of a massless field is calculated in the various proposed vacuum states in several two-dimensional space–times. The results illuminate the physical significance of the states, confirming the interpretations already given to them on *a priori* grounds. These calculations continue a programme in which the quantum energy-momentum tensor has been defined by a covariant point-splitting method and, for two-dimensional massless fields and other simple models, evaluated explicitly with the aid of the solution in normal modes allowed by conformal invariance of the field equation (De Witt 1975, Christensen 1975, 1976, Fulling and Davies 1976, Davies and Fulling 1977a, b, Davies *et al* 1976, 1977, Davies 1976, Fulling 1977, Hiscock 1977, Davies and Unruh 1977, Unruh 1977, Bunch and Davies 1977).

Hurried readers should note that the ratio of physics to technical formalism increases as the paper proceeds. The last section is a summary in the most physical terms.

Conventions: $\hbar = c = 1$; $g_{00} > 0$; $R > 0$ in the spatially closed two-dimensional de Sitter universe.

2. Boost-like Killing vectors and the Kruskal–Rindler transformation

2.1. *Flat and other ultrastatic space–times*

Let us first consider space–time metrics of the form

$$ds^2 = dt^2 - dx^2 - d\Omega^2 \tag{2.1}$$

where $d\Omega^2$ is a two-dimensional positive-definite metric, independent of t and x. The simplest and most important special cases are

$$d\Omega^2 = (dy^1)^2 + (dy^2)^2 \tag{2.2}$$

(yielding Minkowski space), and

$$d\Omega^2 = L_0^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.3}$$

where L_0 is a length. The two-dimensional flat model (without a $d\Omega^2$) is also of interest.

The geometry (2.1) is globally static, ∂_t being the time-like Killing vector. Furthermore, the norm of the Killing vector is constant ($g_{00} = 1$); let us call a space-time of this sort *ultrastatic*.

Define null coordinates

$$V = t + x, \qquad U = t - x, \tag{2.4}$$

so that $dt^2 - dx^2 = dV\,dU$. The (t, x) plane (figure 1) is divided by the lines $V = 0$ and $U = 0$ into four parts, which will be labelled F($V > 0$, $U > 0$), R($V > 0$, $U < 0$), P($V < 0$,

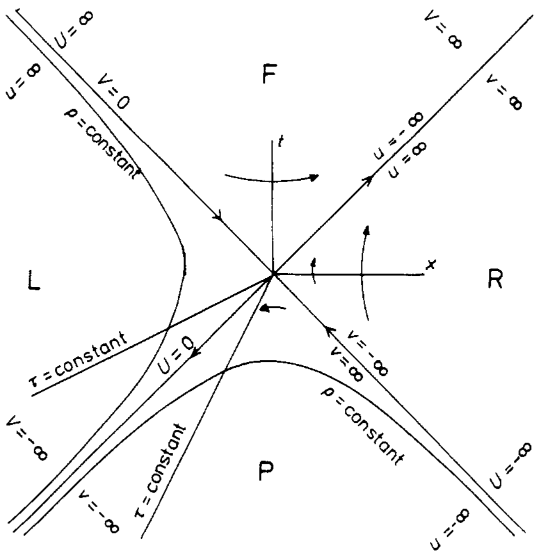

Figure 1. Coordinate systems associated with a boost-like Killing vector, which maps the (t, x) plane as indicated by the arrows.

$U < 0$), and L($V < 0$, $U > 0$). In each quadrant introduce new null coordinates by

$$V = \sigma\, e^{\sigma v}, \qquad U = \pm e^{\pm u}, \tag{2.5}$$

so that

$$v = \sigma \ln|V|, \qquad u = \pm \ln|U|, \tag{2.6}$$

$$dV\,dU = e^{\sigma v \pm u}\, dv\, du = |VU|\, dv\, du, \tag{2.7}$$

where $\pm$ is the sign of U, and σ is the sign of V. Finally, let

$$\sigma v = \tau + \rho, \qquad \mp u = \tau - \rho, \tag{2.8}$$

so that

$$\rho = \tfrac{1}{2}(\sigma v \pm u) = \ln|VU|^{1/2} = \ln|t^2 - x^2|^{1/2}, \tag{2.9}$$

$$\tau = \tfrac{1}{2}(\sigma v \mp u) = \ln|V/U|^{1/2} = \begin{cases} \tanh^{-1}(x/t) & \text{in F and P,} \\ \tanh^{-1}(t/x) & \text{in R and L,} \end{cases} \tag{2.10}$$

and therefore (cf Rindler 1966, Kruskal 1960)

$$\mathrm{d}t^2-\mathrm{d}x^2=\mathrm{d}V\,\mathrm{d}U=\mathrm{e}^{2\rho}\,\mathrm{d}v\,\mathrm{d}u=\mp\sigma\mathrm{e}^{2\rho}(\mathrm{d}\tau^2-\mathrm{d}\rho^2). \tag{2.11}$$

Thus in R and L, where $\mp\sigma=+1$, τ is a time coordinate and ρ a space coordinate, while the reverse is true in F and P. In L and P the direction of these 'times' is the opposite of the natural one. The inverse transformation is

$$t=\mp\mathrm{e}^{\rho}\sinh\tau, \qquad x=\mp\mathrm{e}^{\rho}\cosh\tau \tag{2.12a}$$

in L and R, respectively, while in F and P it is

$$t=\pm\mathrm{e}^{\rho}\cosh\tau, \qquad x=\pm\mathrm{e}^{\rho}\sinh\tau. \tag{2.12b}$$

The metric (2.1) *cum* (2.11) is independent of τ. Thus the (τ,ρ) coordinate system makes explicit the existence of another Killing vector, ∂_τ, which generates a group of transformations mapping the hyperbolic surfaces (ρ = constant) into themselves and the radial surfaces (τ = constant) onto each other in the directions indicated by the arrows in figure 1. These are boosts (homogeneous Lorentz transformations) of the (t,x) plane. The Killing vector is time-like in regions R and L, space-like in F and P, and null on the surfaces $V=0$ and $U=0$ (where $\rho=-\infty$ and $\tau=\pm\infty$).

Clearly, ρ may be replaced in each quadrant by an arbitrary monotonic function of ρ without changing the essential character of the coordinate system. One reasonable alternative is $z=\mathrm{e}^{\rho}$,

$$\mathrm{d}t^2-\mathrm{d}x^2=\mp\sigma(z^2\,\mathrm{d}\tau^2-\mathrm{d}z^2); \tag{2.13}$$

z is the proper distance of the point from the origin, and the (τ,z) system is the counterpart, for an indefinite metric, of ordinary plane polar coordinates. Another is $r=\mp\frac{1}{4}\sigma\,\mathrm{e}^{2\rho}$,

$$\mathrm{d}t^2-\mathrm{d}x^2=4r\,\mathrm{d}\tau^2-r^{-1}\,\mathrm{d}r^2, \tag{2.14}$$

which, with $t^*=2\tau$, maximizes the resemblance to the Schwarzschild metric (2.15).

2.2. *Schwarzschild–Kruskal and more general space–times*

The Schwarzschild metric is ordinarily written

$$\mathrm{d}s^2=\left(1-\frac{2M}{r}\right)\mathrm{d}t^{*2}-\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2-r^2\,\mathrm{d}\Omega^2, \tag{2.15}$$

where $\mathrm{d}\Omega^2$ is the spherical line element (2.3) with $L_0=1$, and M is a length equal to the conventional mass times the gravitational constant. There is a coordinate singularity at $r=2M$, inside which r becomes a time coordinate. The metric in the radius–time plane is made manifestly conformally flat by the transformation

$$r^*=r+2M\ln|1-r/2M|, \tag{2.16}$$

$$\frac{\mathrm{d}r}{\mathrm{d}r^*}=1-\frac{2M}{r}, \tag{2.17}$$

$$\mathrm{d}s^2=\left(1-\frac{2M}{r}\right)(\mathrm{d}t^{*2}-\mathrm{d}r^{*2})-r^2\,\mathrm{d}\Omega^2. \tag{2.18}$$

In analogy to equations (2.5) and (2.8), let

$$V = 4M\sigma \exp[(t^* + r^*)/4M - \tfrac{1}{2}], \qquad U = \pm 4M \exp[(r^* - t^*)/4M - \tfrac{1}{2}]. \tag{2.19}$$

Then we have

$$ds^2 = 2Mr^{-1} e^{1-r/2M} \, dV \, dU - r^2 \, d\Omega^2, \tag{2.20}$$

so that t and x, defined by equations (2.4), are $4M\,e^{-1/2}$ times the coordinates introduced by Kruskal (1960).

More generally, consider any static metric of the form

$$ds^2 = z^2 A_R(z)^2 \, d\tau^2 - dz^2 - B_R(z)^2 \, d\Omega^2 \qquad (z \geqslant 0), \tag{2.21}$$

where $A_R(0) = 1$ and $B_R(0) \neq 0$. The two-dimensional metric $d\Omega^2$ is arbitrary (or absent) as in equation (2.1). If B_R is a monotonic function of z, the coordinate $r \equiv B_R(z)$ may be preferred to z in some contexts (cf equation (2.15)).

The first step in finding the analogue of Kruskal's transformation for this geometry is to put the metric of the (τ, z) plane into manifestly conformally flat form, by adopting a new spatial coordinate, ρ. One must have

$$\rho = \int (zA_R(z))^{-1} \, dz = \ln z + \text{constant} + \ldots, \tag{2.22}$$

where the dots indicate a term which vanishes as $z \to 0$. Let us take the constant of integration as 0, so that

$$ds^2 = e^{2\rho} F_R(-e^{2\rho})^2 (d\tau^2 - d\rho^2) - B_R^2 \, d\Omega^2 \tag{2.23}$$

with $F_R(0) = 1$.

To the metric (2.23) we apply the Kruskal–Rindler transformation in precisely the form (2.4)–(2.12), with the signs appropriate to the region R, and obtain

$$ds^2 + B_R^2 \, d\Omega^2 = F_R(+VU)^2 \, dV \, dU = F_R(t^2 - x^2)^2 (dt^2 - dx^2) \tag{2.24}$$

where $U < 0 < V$ $(x > |t|)$. This space–time can be extended by attaching in the other three quadrants similar τ-independent metrics, defined by functions F_F, F_P, and F_L, all taking the value 1 at $e^{2\rho} = 0$, and B_F, B_P, and B_L. If F_R and B_R are analytic functions of $e^{2\rho}$ near 0, then the unique extension which is analytic in the coordinates (t, x) or (V, U) is given by

$$\begin{aligned} F_F(e^{2\rho}) &= F_P(e^{2\rho}) = F_R(+e^{2\rho}) \equiv F(VU), \\ F_L(-e^{2\rho}) &= F_R(-e^{2\rho}) \equiv F(VU), \end{aligned} \tag{2.25}$$

and similar conditions on the B's.

In all this discussion the possibility is not excluded that the space–time in a quadrant terminates at a genuine singularity at some finite value of ρ or z, as in the Schwarzschild case.

The projection onto the (t, x) plane is as depicted in figure 1, with possible hyperbolic boundaries as in the familiar Kruskal diagram. The *horizon* is the null surface defined by any of the equations $z = 0$, $\rho = -\infty$, $VU = 0$, or, in the Schwarzschild case, $r = 2M$. On the horizon the norm of the Killing vector ∂_τ vanishes; this is the extreme opposite of the ultrastatic situation considered earlier.

We shall call (t, x) or (V, U) *Kruskal coordinates*, while (v, u), (τ, ρ), (τ, z), or (τ, r) will be called *Rindler coordinates*. Both are essentially unique. (The normalization conventions imposed in writing equations (2.21), (2.23), and (2.24) have fixed the units of t and x and of τ and ρ and also the additive constant in ρ. Because of the symmetry, the additive constant in τ and the Lorentz frame in (t, x) space cannot be specified absolutely; the choice (as unity) of constants in equations (2.5) relates these, however, so that $\tau = 0$ on the t and x axes.) It is clear that the Schwarzschild–Kruskal metric fits into this general framework with

$$\tau = t^*/4M, \qquad \rho = r^*/4M + \ln 4M - \tfrac{1}{2}. \tag{2.26}$$

Remark: The arguments and values of transcendental functions (exp, ln, etc) are customarily required to be dimensionless. In the present context this can be achieved by always associating a scale factor L, with the dimension of length, with the coordinates which measure lengths or times. L may be a characteristic length of the geometry, such as $4M$ in the Schwarzschild case (see equations (2.26) and (2.19)). On the other hand, L could be some conventional unit independent of the geometry, such as the metre or the Compton wavelength of the proton. The uncluttered general formulae of this section correspond to this second convention, with units chosen so that $L = 1$ numerically. That convention allows one to treat flat and curved space–times in a unified formalism, and to study the adiabatic limit of a curved geometry (Parker and Fulling 1974, Fulling and Parker 1974, § 6).

2.3. *de Sitter space–times and double black holes*

As an application of the foregoing formalism we consider the n-dimensional closed de Sitter universe of radius r. This can be defined as the manifold of points in $(n+1)$-dimensional Minkowski space satisfying

$$(x^0)^2 - \sum_{j=1}^{n} (x^j)^2 = -r^2. \tag{2.27}$$

(In this connection r will always be a constant, not a coordinate.) Introducing locally static coordinates in, for example, the four-dimensional case by

$$\begin{aligned} &x^0 = r \sinh(\hat{\tau}/r)\cos(\hat{z}/r), && x^1 = r \sin(\hat{z}/r)\sin\theta\cos\phi, \\ &x^2 = r\sin(\hat{z}/r)\sin\theta\sin\phi, && x^3 = r\sin(\hat{z}/r)\cos\theta, \\ &x^4 = r\cosh(\hat{\tau}/r)\cos(\hat{z}/r), && \end{aligned} \tag{2.28}$$

with $0 \leqslant \hat{z} \leqslant \pi r$, one finds that the region $|x^0| < |x^n|$ (equivalently, $(x^1)^2 + \ldots + (x^{n-1})^2 < r^2$) is covered, and that the induced metric is

$$\begin{aligned} ds^2 &= \cos^2(\hat{z}/r)\, d\hat{\tau}^2 - d\hat{z}^2 - r^2 \sin^2(\hat{z}/r)\, d\Omega^2 \\ &= [1 - (\hat{r}/r)^2]\, d\hat{\tau}^2 - [1 - (\hat{r}/r)^2]^{-1}\, d\hat{r}^2 - \hat{r}^2\, d\Omega^2 \end{aligned} \tag{2.29}$$

where $\hat{r} = r\sin(\hat{z}/r)$ and $d\Omega^2$ is the $(n-2)$-dimensional spherical line element. Near the centre ($\hat{z} = \hat{r} = 0$) and the antipode ($\hat{z} = \pi r$, $\hat{r} = 0$ again) the picture is Cartesian to first order, but there is an horizon at $\hat{z} = \pi r/2$ ($\hat{r} = r$). To cast the metric into the form (2.21), take

$$z = |\hat{z} - \tfrac{1}{2}\pi r|, \qquad \tau = \hat{\tau}/r, \qquad zA_{\mathrm{R}}(z) = r\sin(z/r). \tag{2.30}$$

For the region $\hat{z} \leqslant \tfrac{1}{2}\pi r$ it follows that

$$\rho = -\mathrm{gd}^{-1}(\hat{z}/r) + \ln 2r = \ln\tan(z/2r) + \ln 2r, \tag{2.31}$$

where gd is the Gudermannian function (Gradshteyn and Ryzhik 1965, § 1.49), and

$$e^{2\rho} F_{\mathrm{R}}(-e^{2\rho})^2 \equiv r^2 \sin^2(z/r) = r^2 \cosh^{-2}(\rho - \ln 2r). \tag{2.32}$$

(*Note*: A superscript -1 denotes a functional inverse, but all other exponents denote algebraic powers.) Applying the Kruskal–Rindler transformation for region R, one finds

$$\mathrm{d}s^2 = \left(1 - \frac{VU}{4r^2}\right)^{-2} \mathrm{d}V\,\mathrm{d}U - r^2 \tanh^2 \ln\left(\frac{|VU|^{1/2}}{2r}\right) \mathrm{d}\Omega^2. \tag{2.33}$$

If $n = 2$, the horizon degenerates to two disjoint pieces, and it is best to regard $\hat{z}$ as a periodic coordinate which covers a whole Cauchy surface as it varies from $-\pi r$ to πr. A proper-distance coordinate ranging from 0 to πr can be defined near each horizon:

$$\begin{aligned} z_1 &= \left|\hat{z} + \tfrac{1}{2}\pi r \pmod{2\pi r}\right|, \\ z_2 &= \pi r - z_1 = \left|\hat{z} - \tfrac{1}{2}\pi r \pmod{2\pi r}\right|. \end{aligned} \tag{2.34}$$

Kruskal coordinates (spatially reflected in the right-hand case) are introduced through

$$\rho_2 = \ln \tan(z_2/2r) + \ln 2r = -\rho_1 + 2 \ln 2r, \tag{2.35}$$

etc (see figure 2). (The subscripts 1 will sometimes be omitted for convenience.) The formulae given above still apply, with the angular variables omitted. In R the Kruskal coordinates for the left horizon are related to those for the right horizon by

$$V_1 = -4r^2 U_2^{-1} > 0, \qquad U_1 = -4r^2 V_2^{-1} < 0. \tag{2.36}$$

If one approaches the right horizon via the second sheet (L), then the equalities (2.36) still hold but the quantities take the opposite signs.

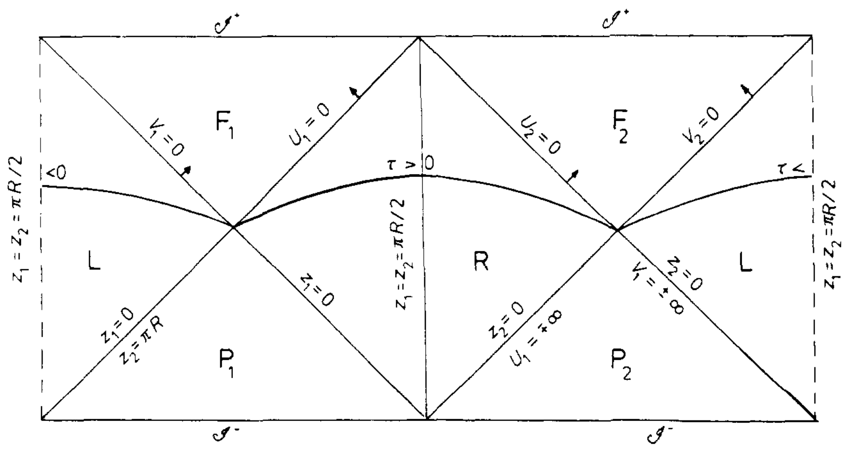

Figure 2. Conformal diagram of two-dimensional de Sitter space–time. The left and right edges are identified. The arrows indicate the directions of increase of the null coordinates that vanish on the associated lines. Note that $U_1 \to -\infty$ as the line $V_2 = 0$ is approached from the upper left, but $U_1 \to +\infty$ as it is approached from the lower right; similarly for V_1 on $U_2 = 0$. Except for the numerical values of z_1 and z_2, the figure applies to any 'double black hole' model.

The analytic extension of the metric (2.33) into the regions 'inside' the left horizon coincides, not surprisingly, with the actual metric of the manifold (2.27) in the domain $x^1 > r$, as may be seen by writing the latter as

$$ds^2 = dz^2 - \sinh^2(z/r)\, d\hat{\tau}^2 \tag{2.37}$$

and applying the Kruskal–Rindler transformation for regions of types P and F. (In this context one has $0 < z = 2r \tanh^{-1}(e^{2\rho}/2r) < \infty$, and hence $0 < e^\rho < 2r$, or $0 < VU = t^2 - x^2 < 4r^2$, not $+\infty$.)

The two-dimensional de Sitter universe has the conformal structure shown in figure 2. The geometry resembles a two-dimensional static closed universe containing two black holes at antipodal points, with their second sheets joined to form another universe of the same kind. In fact, the analogy can be made stronger by replacing the de Sitter metric in the regions $F_{1,2}$ and $P_{1,2}$ by the interior Kruskal metric (first term in equation (2.20)), with its singularity. If the parameters of the two metrics are related by

$$r = 2M, \tag{2.38}$$

then the metric coefficient $F(VU)$ of this model is differentiable at the horizon. (This equivalence of cosmological and black-hole horizons does not extend to four dimensions (cf Gibbons and Hawking 1977).)

We shall see that the de Sitter space–time, as a model of two black holes interacting with a quantum field, is rather trivial. More general two-dimensional static models with the same horizon structure are easy to study. In what will be the region R, let

$$ds^2 = W(z)^2\, d\hat{\tau}^2 - dz^2 \qquad (0 < z < a) \tag{2.39}$$

(where the scale of $\hat{\tau}$ is arbitrary at first), and assume that

$$W(z) \approx \begin{cases} \dfrac{1}{C_1}\left[\dfrac{z_1}{r_1} - \dfrac{1}{6}\left(\dfrac{z_1}{r_1}\right)^3\right] & \text{as } z_1 \equiv z \to 0, \\[2ex] \dfrac{1}{C_2}\left[\dfrac{z_2}{r_2} - \dfrac{1}{6}\left(\dfrac{z_2}{r_2}\right)^3\right] & \text{as } z_2 \equiv a - z \to 0 \end{cases} \tag{2.40}$$

$(C_j, r_j > 0)$. Then, following the general procedure, we take

$$\hat{\tau} = C_j r_j \tau_j, \qquad c = C_2 r_2 / C_1 r_1, \tag{2.41}$$

$$\frac{d\rho_1}{dz_1} = (C_1 r_1 W(z))^{-1} = c(C_2 r_2 W(z))^{-1} = c\frac{d\rho_2}{dz_2} = -c\frac{d\rho_2}{dz_1},$$

so that, for some constant $D > 0$,

$$\rho_1 = -c\rho_2 + \ln D \tag{2.42}$$

(exactly, for *all* z). It follows that

$$-V_1 U_1 = e^{2\rho_1} = D^2 e^{-2c\rho_2} = D^2(-V_2 U_2)^{-c},$$

$$-V_1/U_1 = e^{2\tau_1} = e^{2c\tau_2} = (-V_2/U_2)^c,$$

or

$$|V_1| = D|U_2|^{-c}, \qquad |U_1| = D|V_2|^{-c}. \tag{2.43}$$

In this form equations (2.43) are valid in the whole analytic extension of the metric, with the second sheets identified as in figure 2.

The parameter c will be crucial in determining the compatibility of requirements of analyticity or positive frequency in the respective Kruskal coordinates of the two horizons. Note that this number is not determined by the intrinsic geometry in the immediate vicinity of each horizon, since the latter is given solely by the relative magnitude of the two leading terms in W; the factor C_j is absorbable locally into the time coordinate. With the notation (2.40), the geometry near $z_j = 0$ is to the lowest non-trivial order that of a de Sitter space of radius r_j or, equivalently, a Schwarzschild black hole of mass $\frac{1}{2}r_j$. Thus one may have two black holes of the same mass, but with $c \neq 1$; this situation may be likened to two equal point charges located in regions of constant but unequal electrostatic potential.

A simple model with $c \neq 1$ is formed by joining halves of de Sitter universes of different radius at their centres:

$$ds^2 = \begin{cases} \cos^2(\hat{z}/r_1)\,d\hat{\tau} - d\hat{z}^2 & (-\frac{1}{2}\pi r_1 < \hat{z} < 0), \\ \cos^2(\hat{z}/r_2)\,d\hat{\tau} - d\hat{z}^2 & (0 < \hat{z} < \frac{1}{2}\pi r_2). \end{cases} \tag{2.44}$$

For this an explicit calculation yields for the formulae (2.43)

$$|V_1| = 2r_1(2r_2)^{r_2/r_1}|U_2|^{-r_2/r_1}, \tag{2.45}$$

etc. For a non-trivial example with $c = 1$ we insert a flat strip into de Sitter space:

$$ds^2 = \begin{cases} d\hat{\tau}^2 - d\hat{z}^2 & (|\hat{z}| < b), \\ \cos^2\left(\dfrac{|\hat{z}| - b}{r}\right) d\hat{\tau}^2 - d\hat{z}^2 & (b < |\hat{z}| < \frac{1}{2}\pi r + b); \end{cases} \tag{2.46}$$

whence

$$|V_1| = 4r^2\,e^{2b/r}|U_2|^{-1}, \tag{2.47}$$

etc. In § 4 the 'vacuum' expectation values of the energy–momentum tensor of a field will be calculated in these two models to demonstrate the physical significance of c.

3. Candidates for vacuum state

3.1. *Field quantization*

With the geometrical notation under control, we can turn to physics. For simplicity we consider an Hermitian scalar field minimally coupled to the gravitational field, with no other external potentials. Mathematically, this is an operator-valued distribution on space–time, whose values are related among themselves in certain ways, expressed by a partial differential equation and by commutation relations. (The commutation relations presuppose a distinction between the forward and backward directions of time, continuous over the space–time manifold. We take the forward direction to agree with the positive t direction in the Kruskal system. This does not prevent us, of course, from using for technical purposes time coordinates, τ and ρ, which 'run backwards' in regions L and P.) The field equation and commutation rules determine, up to mathematical niceties, an abstract algebra of field 'operators', but the realization of the algebra as concrete operators on a Hilbert space is not unique.

The following (which can be stated more abstractly (Ashtekar and Magnon 1975)) is the most common and convenient way to build such a representation in the case of an

Hermitian field satisfying a globally hyperbolic linear equation. In a general solution of the field equation,

$$\phi(t, \boldsymbol{x}) = \int d\mu(j)(\phi_j(t, \boldsymbol{x})a_j + \phi_j^*(t, \boldsymbol{x})a_j^\dagger), \tag{3.1}$$

interpret the coefficients a_j and $a_j^\dagger$ as operators. (The solutions ϕ_j are orthonormalized so that the field commutation relations are equivalent to $[a_j, a_{j'}] = 0$, $[a_j, a_{j'}^\dagger] = \delta(j, j')$, the latter being the Dirac delta distribution with respect to the measure μ. In particular, ϕ_j is a *solution of positive norm*—that is,

$$(\phi_j, \phi_{j'}) \equiv \mathrm{i}\int d\boldsymbol{x}(-g)^{1/2}g^{0\mu}(\phi_j^*\partial_\mu\phi_{j'} - \phi_j\partial_\mu\phi_{j'}^*) = +\delta(j, j'), \tag{3.2}$$

and ϕ_j^* has negative norm.) Postulate a 'vacuum' vector defined (up to phase) by the property $a_j|0\rangle = 0$ for all j. Then the Hilbert space is generated by acting formally on $|0\rangle$ with the elements of the field algebra. The set of ϕ_j is far from unique; different choices may yield the same vacuum vector, different vectors in the same abstract Hilbert space, or unitarily inequivalent representations.

Two points of view may be adopted towards this construction with respect to physical interpretation. According to the first, the a_j and $a_j^\dagger$ are to be annihilation and creation operators for physical particles, and the vacuum is the state of the system in which no particles are present at some particular time (which may be $-\infty$). The proper choice of the ϕ_j must be made on the basis of a careful analysis of the physical meaning of the theory, perhaps by studying the interaction of the field with particle detectors (cf Unruh 1976). It is by no means clear that this kind of interpretation is possible under the most general conditions.

The alternative approach is to forswear particle language and to treat the field aspect of nature as the fundamental one, operationally as well as formally. Then any apparatus of creation and annihilation operators is seen as just one way of labelling states of the field algebra, much like an arbitrarily chosen coordinate system in a curved manifold. Unfortunately, this philosophy does not offer relief from the responsibility to choose a 'good' vacuum state in practice. The most promising avenue to contact between field theory and empirical physics is through *currents* (including the energy–momentum tensor), which are, formally, polynomials in the field and its derivatives. There is evidence (e.g. Parker and Fulling 1974) that these objects can be given meaning (as operator-valued distributions) by a plausible renormalization procedure only in certain 'physical' representations of the field algebra. Thus there is the problem of deciding which vacuum states, if any, belong to physical representations. Apart from this, to see the physical meaning of a vacuum state defined by an expansion (3.1), from this point of view, one must calculate the expectation value in that state of all the relevant currents. On the other hand, if one does hold to a particle interpretation of the theory, then one expects these expectation values to confirm that particle picture, since presumably the particles carry quantities such as energy and charge.

If the space–time metric is static, then a natural choice of the basis solutions, ϕ_j and ϕ_j^*, is those which are eigenfunctions of the time-like Killing vector, ∂_τ. Their time dependence is of the form $e^{-i\omega_j\tau}$, where $\omega_j > 0$ for a ϕ_j and $\omega_j < 0$ for a ϕ_j^*. The corresponding vacuum state is the ground state of a 'Hamiltonian' which generates τ-translations. It does not follow, however, that this state has the full physical significance of the vacuum in special-relativistic field theories. In particular, if the space–time manifold terminates at a horizon beyond which it can be continued, and if the extended manifold has a time-like Killing vector ∂_t, then the vacuum relative to ∂_t may be a different state from the vacuum relative to ∂_τ—even if attention is confined to the field operators for the interior of the original manifold (Fulling 1973). Obviously, this phenomenon must be understood before the true significance of the ∂_τ-static vacuum can be appreciated in cases, such as the Schwarzschild metric, where the extended manifold is not globally static. The major clarification of such questions recently achieved by Unruh (1976) will be elaborated here in the three contexts introduced in § 2.

3.2. *The globally ultrastatic case*

The solutions of the scalar field equation of mass m in a space–time with the ultrastatic metric (2.1), with $-\infty < x < \infty$, can be expanded in the generalized eigenfunctions

$$(4\pi E)^{-1/2}\, e^{-iEt}\, e^{ipx}\psi_j, \qquad E^2 = q_j^2 + p^2 + m^2, \tag{3.3}$$

where

$$\Delta_\Omega \psi_j = -q_j^2 \psi_j; \tag{3.4}$$

Δ_Ω is the Laplace–Beltrami operator (assumed self-adjoint) on the manifold with metric $d\Omega^2$. The mode index j stands for a pair, $p = p_j$ and ψ_j, the latter a member of an orthonormal basis of eigenfunctions of Δ_Ω. In the case of flat space (2.2), the quantization (3.1) results in the conventional Poincaré-invariant vacuum if the ϕ_j are the solutions (3.3) with $E = E_j > 0$, and hence the ϕ_j^* are those with $E < 0$. The state defined in the same way for an arbitrary *ultra*static geometry is surely the 'physical vacuum state' for such a system if anything is. Even the most cautious skeptic will admit that it is a state of great interest, against which others can fruitfully be compared.

Since $E \pm p$ always has the same sign as E (or vanishes), the plane waves (3.3) with $E > 0$ are analytic and bounded in the domain (di Sessa 1974)

$$\operatorname{Im} t < 0, \qquad |\operatorname{Im} x| < |\operatorname{Im} t|. \tag{3.5}$$

Equivalently (Unruh 1976), they are analytic and bounded in the lower half-planes of the variables V and U. These properties are shared by any normalizable solution formed by integrating the positive-frequency plane waves over a square-integrable coefficient function, $f(p)$. On the other hand, a solution containing a negative-frequency component cannot be bounded in the lower half V and U planes (and need not be analytic there or on the real axes themselves). Thus we have a criterion for identifying purely positive-frequency solutions.

Now consider the same field equation written in the coordinates (τ, z) of equation (2.13). (Recall that $\pm\sigma$ is -1 in regions R and L, $+1$ in F and P.) The normal-mode solutions in each quadrant are proportional to

$$e^{-i\omega\tau}\chi_j(z)\psi_j, \tag{3.6}$$

where

$$z\partial_z(z\,\partial_z\chi_j) \pm \sigma z^2(q_j^2 + m^2)\chi_j + \omega^2\chi_j = 0. \tag{3.7}$$

This equation can be solved exactly.

Singular case $(q_j^2 + m^2 = 0)$:

$$\chi_j = e^{ik\rho}, \qquad \rho = \ln z, \qquad k^2 = \omega^2. \tag{3.8}$$

Generic case:

$$\chi_j = Z_{i\omega}(K_j z), \qquad K_j = (q_j^2 + m^2)^{1/2}, \tag{3.9}$$

$Z_{i\omega}$ is a Bessel function of imaginary order if one is considering F or P, but a modified Bessel function of imaginary order in R or L.

In F, where τ is a spatial coordinate, a basis set of positive-norm solutions ϕ_j for use in equation (3.1) will include all values (positive and negative) of ω, which is an ordinary Fourier transform variable on each hyperbola of constant time ρ; the index j stands for ψ_j and ω. There is, *a priori*, an infinite family of possible choices of the $Z_{i\omega}$ for these ϕ_j. However, if we require that the vacuum state be the 'physical' one associated with the global time-like Killing vector, then $Z_{i\omega}$ must be proportional to $H^{(2)}_{i\omega}$, since only that Bessel function has the proper behaviour in the complex domain (3.5) (di Sessa 1974). That is, the normalizable superpositions (wave packets) of the solutions

$$e^{-i\omega t}\, e^{\pi\omega/2} H^{(2)}_{i\omega}(Kz)\psi_j = e^{-i\omega\tau}\, e^{-\pi\omega/2} H^{(2)}_{-i\omega}(Kz)\psi_j \tag{3.10}$$

are precisely the normalizable superpositions of the plane waves $e^{-iEt}\,e^{ipx}$ with $E = +(p^2+K^2)^{1/2}$. This conclusion also follows from an explicit integral representation of the Hankel function (cf Fulling *et al* 1975, § 4, Sommerfield 1974). In P the positive-frequency functions involve $H^{(1)}_{i\omega}$ instead of $H^{(2)}_{i\omega}$.

Turning to the more interesting quadrant R, we see that a complete set of functions in the $L^2(\rho)$ space on a ray of constant time τ is given by $e^{ik\rho}$ $(-\infty < k < \infty)$ in the singular case, and by $K_{i|\omega|}\,(Ke^{\rho})$ in the generic case (Titchmarsh 1962, § 4.15). In the former case, waves with $k > 0$ propagate from the (past) horizon out to infinity, and those with $k < 0$ travel in from infinity to the future horizon. Wave packets formed from the $K_{i|\omega|}$, however, propagate out of the horizon in the past and are reflected back over the horizon in the future. (Any linearly independent solution, such as $I_{i|\omega|}$, blows up at infinity.) The two cases correspond respectively to classical null trajectories (which reach $\mathscr{I}^+$ or $\mathscr{I}^-$) and to trajectories time-like in the (t, x) plane (which in general both enter and leave R via the horizon).

If we regard R as a static universe with Killing vector ∂_τ, then the natural basis solutions ϕ_j are those with $\omega > 0$ in equation (3.6), χ_j having the space dependence just described. Now a τ ray in R and a τ ray in L together constitute a Cauchy surface for the whole space–time manifold. (Special attention must be given to the origin if distributions are allowed as initial data.) The general solution of the field equations, therefore, is a superposition of these ϕ_j and ϕ_j^* and the analogous functions in L. (In L the positive-norm solutions are those with $\omega < 0$.) So we have the normalized basis solutions ($\sigma = +1$ in R, -1 in L)

$$\phi_j = \phi_{k\sigma} = \theta(\sigma x)(4\pi|k|)^{-1/2}\, e^{-i\sigma|k|\tau}\, e^{ik\rho}\psi_j, \qquad j = (\sigma, k, \psi_j), -\infty < k < \infty \tag{3.11}$$

in the singular case, and

$$\phi_j = \phi_{\gamma\sigma} = \theta(\sigma x)\pi^{-1}\sinh^{1/2}(\pi\gamma)\, e^{-i\sigma\gamma\tau} K_{i\gamma}(K_j z)\psi_j, \qquad j = (\sigma, \gamma, \psi_j), 0 < \gamma < \infty \tag{3.12}$$

in the generic case, along with their negative-norm conjugates. If ψ_j is not real, it must be replaced by ψ_j^* when $\sigma = -1$.

The solutions $\phi_{\gamma\sigma}$ are not superpositions of the positive-frequency plane waves (3.3). That distinction belongs to the combinations

$$\begin{aligned} \hat{\phi}_\gamma &= [2\sinh(\pi\gamma)]^{-1/2}(e^{\pi\gamma/2}\phi_{\gamma+} + e^{-\pi\gamma/2}\phi^*_{\gamma-}), \\ \hat{\phi}_{-\gamma} &= [2\sinh(\pi\gamma)]^{-1/2}(e^{-\pi/2}\phi^*_{\gamma+} + e^{\pi\gamma/2}\phi_{\gamma-}). \end{aligned} \tag{3.13}$$

(Note that the normalization (3.2) is preserved, and that $\hat{\phi}_\gamma$ is concentrated mostly in R, while $\hat{\phi}_{-\gamma}$ is larger in L.) Indeed, $\hat{\phi}_{\pm\gamma}$ propagates into F as

$$\hat{\phi}_\omega = -\mathrm{i}\, 8^{-1/2}\, \mathrm{e}^{-\mathrm{i}\omega\tau}\, \mathrm{e}^{-\pi\omega/2} H^{(2)}_{-\mathrm{i}\omega}(K_j z)\psi_j \qquad (-\infty < \omega = \pm\gamma < \infty) \quad (3.14)$$

(cf equation (3.10)). This was proved by di Sessa (1974) by analytic continuation from F into L and R through the complex domain (3.4). (See also Sommerfield (1974). A more direct method would be to match wave packet solutions at the horizon using Cartesian coordinates, following Boulware (1975).) Unruh, however, established equations (3.13) without considering region F at all, simply by requiring that $\hat{\phi}_\omega$ be analytic and bounded in the lower half U plane when restricted to the horizon surface $V = 0$ (or vice versa). This is the prescription which generalizes to models without a global Killing vector.

The Bogolubov transformation between the $\phi_{\gamma\sigma}$ modes and the plane-wave modes (see Fulling 1973) has hereby been factored into two transformations: a unitary one relating eigenfunctions of ∂_τ to eigenfunctions of ∂_t and ∂_x and expressed by the Fourier integral representation of the Hankel functions, and a 'diagonal' one given by equations (3.13). The latter clearly does not have a Hilbert–Schmidt kernel, since γ is a continuous variable. Therefore, the Fock representations built on the ∂_t vacuum and on the ∂_τ vacuum are not unitarily equivalent.

The situation in the singular case is similar. For each value of $|k|$ there are now four solutions of positive frequency with respect to t, which are obtained from those branches of the functions $V^{\mathrm{i}k}$ and $U^{\mathrm{i}k}$ (k positive or negative) which are analytic in the lower half-planes (that is, in evaluating the complex powers, -1 must be interpreted as $\mathrm{e}^{-\mathrm{i}\pi}$ if $+1$ is taken as e^0). The normalized solutions are, in terms of the functions (3.11),

$$\begin{aligned} \hat{\phi}_{k(+)} &= [2\sinh(\pi|k|)]^{-1/2}(\mathrm{e}^{\pi|k|/2}\phi_{k+} + \mathrm{e}^{-\pi|k|/2}\phi^*_{-k-}) \\ &= \begin{cases} N_k U^{\mathrm{i}k}\psi_j & (k>0), \\ N_k V^{\mathrm{i}k}\psi_j & (k<0), \end{cases} \end{aligned} \qquad (3.15a)$$

$$\begin{aligned} \hat{\phi}_{k(-)} &= [2\sinh(\pi|k|)]^{-1/2}(\mathrm{e}^{-\pi|k|/2}\phi^*_{-k+} + \mathrm{e}^{\pi|k|/2}\phi_{k-}) \\ &= \begin{cases} N_k V^{\mathrm{i}k}\psi_j^* & (k>0), \\ N_k U^{\mathrm{i}k}\psi_j^* & (k<0), \end{cases} \end{aligned} \qquad (3.15b)$$

where

$$N_k = [8\pi|k|\sinh(\pi|k|)]^{-1/2}\,\mathrm{e}^{-\pi k/2}. \qquad (3.16)$$

Here k is the 'momentum' in region R; that is, all the functions go as $\mathrm{e}^{+\mathrm{i}k\rho}$ there. However, the direction of motion of wave packets depends on the relative sign of k and the frequency with respect to τ; U waves propagate to the right, V waves to the left in x space. The functions $\hat{\phi}_{k(+)}$ are larger in R than in L, and vice versa for $\hat{\phi}_{k(-)}$. The conjugate basis functions $\hat{\phi}^*_{k(\sigma)}$, negative in t frequency and in norm, are boundary values of branches of the power functions analytic in the upper half-planes.

Remark: In the singular case, initial data on a constant-time hyperbola in F do not determine a unique solution throughout space–time (contrast equation (3.14)). Only the U-wave part in L and the V-wave part in R are determined. The other components are related to freely specifiable data on a hyperbola in P.

3.3. *The general case—one horizon*

Consider a space–time with a metric of the type (2.23)–(2.24), analytically extended to all quadrants as in equations (2.25) (see also equation (2.11)). In particular, the geometry of L is identical to that of R, reflected. We may now omit the subscript R on B and F. In the minimally coupled scalar field equation, separate variables in the Rindler system, writing

$$\phi_j = e^{-i\omega_j\tau} B(\rho)^{-1} \chi_j(\rho)\psi_j, \tag{3.17}$$

where the ψ_j, as before, are the normalized eigenfunctions (e.g., spherical harmonics) in the two irrelevant coordinates, with eigenvalues q_j^2 (equation (3.4)). One finds, as the generalization of equation (3.7),

$$-\partial_\rho^2\chi_j + B^{-1}(\partial_\rho^2 B)\chi_j \mp \sigma\, e^{2\rho}F^2(B^{-2}q_j^2 + m^2)\chi_j = \omega_j^2\chi_j. \tag{3.18}$$

(B^{-1} has been factored out of χ in order to free equation (3.18) of first derivatives.)

Let us discuss the spectra of independent solutions in the various regions, starting with P and F. There the geometry is homogeneous in the spatial coordinate τ, but is not static; the metric has the mathematical form characteristic of simple model cosmologies (cf Kantowski and Sachs 1966). Here ω is an ordinary Fourier transform variable, and for each value of $\omega(-\infty<\omega<\infty)$ there are two linearly independent solutions of equation (3.18), which can be chosen in accordance with the standard complex conjugation and orthonormalization conventions described in § 3.1. In general (unlike the static case discussed in § 3.2) there is no way to make this choice unique, although the arbitrariness can be cut down by requiring, in the limit of large frequency $|\omega|$, a 'positive-frequency' behaviour in a generalized WKB approximation to χ_j (Parker and Fulling 1974).

In what follows, a number of precise notions of positive frequency will be set forth in terms of the behaviour of the functions on the horizon. Each of these determines a definite quantum state of the field inside the horizon as well as outside. It will not be necessary, however, to consider the regions P and F further in order to give a complete mathematical and physical treatment of the field in the static regions outside the horizon (R and L).

Outside the horizon one expects to expand the field in generalized eigenfunctions of the Schrödinger-type operator on the left-hand side of equation (3.18), with $\mp\sigma = 1$. The analysis is an exercise in one-dimensional quantum scattering theory (e.g., Messiah 1961, chap. 3). Barring pathological behaviour of the functions F and B in the positive-ρ direction (away from the horizon), the operator will be self-adjoint. General arguments then show that it is positive. (Change to the variable $B^{-1}\chi$, and integrate by parts the inner product in $L^2(B^2\, d\rho)$.) Furthermore, in most cases of interest F and B will be analytic functions of $e^{2\rho}$ at the horizon (see § 2.2); this implies that the coefficients $B^{-1}\,\partial_\rho^2 B$ and $e^{2\rho}F^2$ vanish as $\rho\to-\infty$. Consequently, the continuous spectrum of ω^2 extends all the way down to 0, and there are no eigenvalues below 0. The spectral multiplicity for a given ω^2 depends on the behaviour of F and B in the positive direction and on the values of q_j^2 and m^2. That is, for some ω (*simple case*) there will be only one well behaved eigenfunction χ_j, going like $e^{i|\omega|\rho} + e^{-i\omega|\rho+i\delta}$ as $\rho\to-\infty$ and falling off rapidly at large ρ, while for other ω (*double case*) there will be two χ, which may be chosen to have 'incoming' behaviour as $\rho\to-\infty$:

$$\vec{\chi} \sim e^{i|\omega|\rho} + R\, e^{-i|\omega|\rho}, \tag{3.19a}$$

$$\overleftarrow{\chi} \sim T\, e^{-i|\omega|\rho}. \tag{3.19b}$$

(A normalization factor has been omitted; the phase is arbitrary but fixed. It is understood that at $\rho = +\infty$, $\overleftarrow{\chi}$ contains a left-moving wave of unit amplitude, and T is the resulting transmission coefficient.) Wave packets formed from $\overrightarrow{\chi}$ functions come up out of the past horizon and are partially transmitted to the right (large ρ) and partially reflected back into the future horizon. Packets of the $\overleftarrow{\chi}$ type are analogous, but come originally from the right. Wave packets containing only simple ω originate in the horizon and are totally reflected. These statements apply only if ω in equation (3.17) is the positive root of ω^2; if ω is negative, the motion is reversed in time. This observation implies (or follows from) the fact that $\overrightarrow{\chi}^*$ and $\overleftarrow{\chi}^*$ form a basis alternative to (3.19), corresponding (for $\omega > 0$) to 'outgoing' wave packets, which have a definite direction of motion in the *future*. (Note that 'incoming' and 'outgoing' refer to the motion relative to the region R, not relative to the horizon. The terminology of Unruh (1976) is different.)

In the generic case in § 3.2 the entire spectrum was simple, whereas the singular case was a degenerate example of a double spectrum, where no reflection occurred and hence there was no distinction between incoming and outgoing packets. For the Schwarzschild metric the spectrum is entirely double if $m = 0$ but otherwise it becomes simple for $0 < |\omega| < m$. (Massive particles of low energy can be gravitationally bound to the black hole.) The eigenfunctions have been studied by various authors (e.g., Persides 1976 and other papers, Boulware 1975, Rowan and Stephenson 1976). Another interesting model is that in which the Schwarzschild metric terminates at a surface of constant r where the field is required to vanish; in other words, the black hole is located at the centre of a perfectly reflecting spherical shell. In this case all ω are simple.

Now we turn to the actual quantization of the scalar field which has the classical normal modes (3.17). We consider only Fock representations based on equation (3.1).

The most obvious thing to do is to use those functions (3.17) for which $\omega > 0$ as the 'positive-frequency' elements in the field expansion (3.1), regarding the state annihilated by the corresponding a_j operators as the 'vacuum'. Thereby the field operator is defined throughout region R. It is trivial to extend this quantization to the other three quadrants, precisely as that was done for the globally static situation in the discussion leading up to equations (3.11) and (3.12). The labels on the basis functions, summarized by j, include $\sigma = \pm 1$ to indicate whether the function is initially concentrated in R or in L, the positive frequency ω, and whatever quantum numbers are needed to label the 'angular' wavefunction ψ_j. In addition, if ω_j^2 is a double spectral value, we need another index, ν, to distinguish between $\overleftarrow{\chi}$ $(\nu = +1)$ and $\overrightarrow{\chi}$ $(\nu = -1)$.

For the special case of the Schwarzschild metric, this definition of the vacuum was probably first discussed explicitly by Blum (1973) and later was studied in detail by Boulware (1975). Unruh (1976) calls it the 'η definition'; in his notation for the mode functions, the sign of σ appears as a subscript and the sign of ν as a superscript, both on the left of the symbol ϕ.

A definition of the vacuum enables one to calculate various quantities of physical interest, such as the Feynman propagator, the vacuum expectation value of the time-ordered product of the field operators at two points. (Boulware (1975) actually worked in the opposite direction, defining the Feynman function as a Green function for the field equation having a certain positive-frequency behaviour, and thence deducing properties of the vacuum state.) The η vacuum yields a propagator with a singularity or cusp on the horizon (see Boulware 1975, Unruh 1976). Furthermore, the function vanishes when one of the two points is in L and the other in R, no matter how small the geodesic distance between them. (A Feynman propagator does not vanish in general when its arguments are space-like separated.) This peculiar behaviour of the η propagator at the horizon does not correspond to any singularity in the local geometry there. Rather, it can be traced to the fact that the boost-like Killing vector, with respect to which 'positive frequency' has been defined, becomes null there. In fact, it is obvious that the η vacuum is a strict analogue of the vacuum state of the representation of the free scalar field in two-dimensional Minkowski space constructed in Fulling (1973) (and § 3.2 above), which is physically and mathematically inequivalent to the standard representation.

As mentioned previously, there are two attitudes one might take toward this situation. If one expects to have a unique vacuum that is somehow fundamental, one might well suspect that the η vacuum is physically unacceptable. For instance, near the apex of the horizon (where the four quadrants meet) the Feynman function and other such quantities are grossly different from their flat-space counterparts, even if space–time is essentially flat near that point (as when the Schwarzschild mass M or de Sitter radius r is very large). The construction of the vacuum is 'coordinate dependent' in the sense that it is based in an essential way on the Killing vector ∂_τ, and the singularity in the vacuum state corresponds to the coordinate singularity on the horizon. On the other hand, one might regard the η vacuum as a possible quantum configuration of the field, but only one among many, of which others may be more plausible physically. (In § 4 it will be shown that the η vacuum contains an infinite build-up of energy at the horizon, which can be eliminated by a less singular choice of boundary conditions on the quantum state.) Either point of view motivates a search for alternative vacuum states.

One alternative is Unruh's 'ξ definition' of the vacuum. This is a generalization of the construction of the ordinary Minkowski-space vacuum out of Rindler normal modes through equations (3.13) or (3.15). Consider a wave packet in R formed from the mode functions ϕ_j of equation (3.17), all χ_j being of the right-moving ($\nu = -1$) type whenever mode doubling occurs. In the notation of equations (3.19), $\vec{\chi}$ appears when $\omega > 0$ and $\vec{\chi}^*$ when $\omega < 0$. Examine the form of this solution near the past horizon ($V = 0$, $U < 0$). There the term involving the reflection coefficient R does not contribute, and the other term of each mode function goes, to lowest order, as

$$\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{e}^{\mathrm{i}\omega\rho} = \mathrm{e}^{-\mathrm{i}\omega u} = (-U)^{\mathrm{i}\omega},$$

where equations (2.8) and (2.6) have been used. The analogy with the flat-space case suggests considering the solutions, defined over both R and L,

$$\begin{aligned} \hat{\phi}_\gamma &= [2\sinh(\pi\gamma)]^{-1/2}[\mathrm{e}^{\pi\gamma/2}({}^{-}_{+}\phi_\gamma) + \mathrm{e}^{-\pi\gamma/2}({}^{-}_{-}\phi_\gamma)^*], \\ \hat{\phi}_{-\gamma} &= [2\sinh(\pi\gamma)]^{-1/2}[\mathrm{e}^{-\pi\gamma/2}({}^{-}_{+}\phi_\gamma)^* + \mathrm{e}^{\pi\gamma/2}({}^{-}_{-}\phi_\gamma)], \end{aligned} \tag{3.20}$$

since these are analytic in the lower half U plane when restricted to the surface $V = 0$ (and have positive norm). Here $\gamma (= |\omega|)$ is always a positive number, and ${}^{-}_{+}\phi_\gamma$ and $({}^{-}_{+}\phi_\gamma)^*$ are mode functions in R of the form just described, for, respectively, positive and negative ω:

$${}^{-}_{+}\phi_\gamma = \mathrm{e}^{-\mathrm{i}\gamma\tau} B(\rho)^{-1} \vec{\chi}(\rho)\psi_j\theta(x). \tag{3.21a}$$

(The ψ_j quantum numbers are suppressed on ϕ and χ; x is the Kruskal coordinate defined in § 2.) Finally, ${}^{-}_{-}\phi_\gamma$ and $({}^{-}_{-}\phi_\gamma)^*$ are the corresponding functions defined in L; these are 'outgoing' since they consist of waves entering L from its past null infinity and past horizon and joining to form a normalized wave going out over the future horizon ($V = 0$, $U > 0$) (see figure 3). The explicit form of $({}^{-}_{-}\phi_\gamma)^*$ is

$$({}^{-}_{-}\phi_\gamma)^* = \mathrm{e}^{-\mathrm{i}\gamma\tau} B(\rho)^{-1} \vec{\chi}(\rho)\psi_j\theta(-x), \tag{3.21b}$$

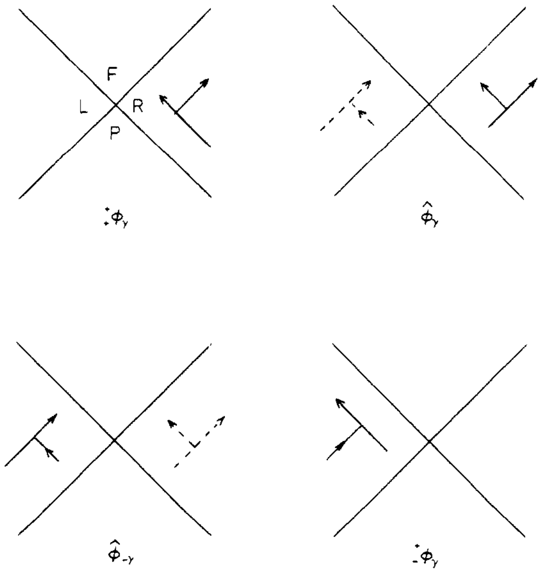

Figure 3. Fluxes associated with the positive-norm basis solutions defining the ξ vacuum. Broken lines indicate negative-norm functions which are exponentially small relative to the accompanying positive-norm parts.

which is a negative-norm function by itself. Compare equations (3.21) with equations (3.11) and (3.12).

Although the analyticity properties provide a convenient technical means to characterize the $\hat{\phi}$ functions, the real justification for choosing these functions as the fundamental 'positive-frequency' basis elements has greater geometrical and physical cogency. Since the components of the metric tensor depend only on $e^{2\rho} \equiv -VU$, the geometry of this space–time in the immediate vicinity of the horizon is everywhere the same, to lowest order, as at the horizon of some globally ultrastatic model (cf equations (2.23)–(2.24) and equations (2.1) and (2.11)). *If a portion of the horizon is used as part of a Cauchy surface, therefore, there should be no physical ambiguity in the notion of vacuum initial conditions on that portion, and the mathematical expression of that condition should take the same form as in the ultrastatic case, which is a trivial extension of the flat case.* The ξ vacuum is based on the tilde-shaped Cauchy surface consisting of the future null infinity of region L, the surface $V = 0$ which is part of the horizon, and the past null infinity of region R. (This statement does not apply literally to a massive field, for which the asymptotic information needed to fix a solution cannot be expressed as data on null infinity. The type of information needed on the horizon is the same as in the massless case, however.) The values taken by a solution on the $V = 0$ surface determine the parts of the solution proportional to normal modes with $\nu = -1$. The natural coordinate on the surface is U, and in fact translation in U is an isometry of the surface, and an approximate isometry of the surrounding neighbourhood. This symmetry reflects the essentially ultrastatic nature of the geometry there. The theory of § 3.2 can now be applied. A complete set of functions on the surface $V = 0$ is the family $e^{-i\lambda U}$, and those with $\lambda > 0$ are the positive-frequency ones. (Recall that when $V = 0$, the solutions (3.3) of the globally ultrastatic model reduce to this form with $\lambda = \frac{1}{2}(E + p)$.) It follows that the positive-frequency solutions in general are those which are analytic in the lower half U plane when $V=0$, and thus one is led to the identification of the $\hat{\phi}_j$ as a basis of ξ-positive-frequency solutions adapted to Rindler coordinates. The corresponding vacuum state is supposed to be one in which, in some sense, there are no particles in existence on $V=0$. This interpretation, suitably qualified, will be buttressed by the energy calculations presented in § 4.

The solutions ${}^{+}_{\pm}\phi_{\pm\omega}$ (formed from $\tilde{\chi}$) form wave packets which vanish at $V=0$, so that analyticity there does not require them to be mixed among themselves like the ${}^{-}_{\pm}\phi_{\pm\omega}$. Indeed, the ξ definition specifies that they be unmixed, so that the ξ-positive-frequency modes are all positive frequency in the usual sense (or vanishing) on the past null infinity ($\mathscr{I}^{-}$) of R (see figure 3). This prescription applies only to models which are asymptotically flat, so that v is an affine parameter on $\mathscr{I}^{-}$.

The ξ definition is asymmetric under time reversal, since it distinguishes the past horizon and past null infinity of R as the surface where the state of the field is to be made vacuous. In general (as will be seen in detail in § 4) there will be radiation crossing the future horizon and null infinity of R. The time-reversed state is obtained by using outgoing basis functions in the construction instead of incoming ones, and studying their behaviour at $U=0$ instead of $V=0$. Furthermore, the time asymmetry of the ξ definition is itself asymmetric with respect to R and L, since the construction uses incoming modes in one and outgoing modes in the other. Related to these features is the asymmetric role of V and U in the construction. We shall now discuss some other vacuum states with different symmetry properties.

First, consider the possibility of requiring analyticity in the lower half V plane when $U=0$, as well as the reverse (cf Hartle and Hawking 1976, Israel 1976, Gibbons and Perry 1977). This is the generalization of the treatment of the 'singular case' of the ultrastatic model (equations (3.15)); let us call it the 'υ definition'. Near the future horizon ($U=0$, $V>0$) only the reflected and transmitted waves in equations (3.19) contribute to wave packets. Since in R

$$e^{-i\omega\tau}\,e^{-i\omega\rho}=e^{-i\omega v}=V^{-i\omega},$$

the analytic continuation proceeds exactly as in the case of $(-U)^{i\omega}$. (The two sign differences compensate.) Therefore, the $\hat{\phi}_\omega(-\infty<\omega<\infty)$ are already analytic in the lower half V plane, and to make the $\nu=+1$ functions satisfy that condition it is necessary only to combine them with the same coefficients as appear in equations (3.20):

$$^{+}\hat{\phi}_\gamma\equiv[2\sinh(\pi\gamma)]^{-1/2}[e^{\pi\gamma/2}({}^{+}_{+}\phi_\gamma)+e^{-\pi\gamma/2}({}^{+}_{-}\phi_\gamma)^{*}],\tag{3.22}$$

and similarly for ${}^{+}\hat{\phi}_{-\gamma}$ (see figure 4). The vacuum state resulting from this construction is independent of the choice of incoming basis functions over outgoing ones (or any others), since all solutions with the same ω are treated alike. The υ vacuum is a state in

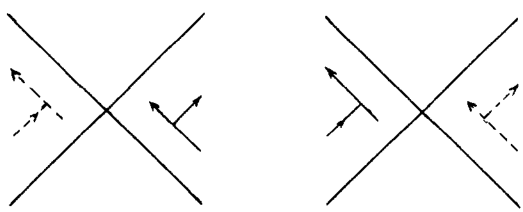

Figure 4. Structure of the basis modes which replace ${}^{+}_{+}\phi_\gamma$ and ${}^{+}_{-}\phi$ (figure 3) in the υ definition.

which space–time is, to the greatest extent possible, empty in the neighbourhood of the entire horizon. However, one would expect in general to find evidence of some kind of excitation of the field at spatial or null infinity (see § 4). In the case of a Schwarzschild black hole inside a reflecting spherical box, where the entire spectrum is simple and there is no spatial infinity, there is no distinction among the υ vacuum, the ξ vacuum, and the time-reversed ξ vacuum. From a physical point of view, anything radiated by a black hole in such a situation must be reflected back into the hole, whereas a black hole in infinite space is free to radiate, to absorb incoming radiation, or to do various coherent combinations of these things.

Returning to the ξ definition, we note that, as far as physics inside the region R is concerned, the important feature of equations (3.20) is the coefficients of the functions with support in R, ${}^{-}_{+}\phi_\gamma$ and $({}^{-}_{+}\phi_\gamma)^*$. (In the η basis the corresponding coefficients are 1 and 0.) Observables defined as *local functions* of the field will not have their expectation values at points in R changed if the L-functions in equations (3.20) are replaced by their time reversals. That is, instead of the function (3.21b) we use

$$\mathrm{e}^{-\mathrm{i}\gamma\tau}B^{-1}\vec{\chi}^*\psi_j\theta(-x).$$

(If preferred, ψ_j could be replaced by ψ_j^*.) Similarly, the basis mode ${}^{+}_{-}\phi_\gamma$ is replaced by its time reversal, which is purely incoming from the past null infinity of L. This will be called the 'λ definition' (see figure 5). These functions cannot be characterized by a

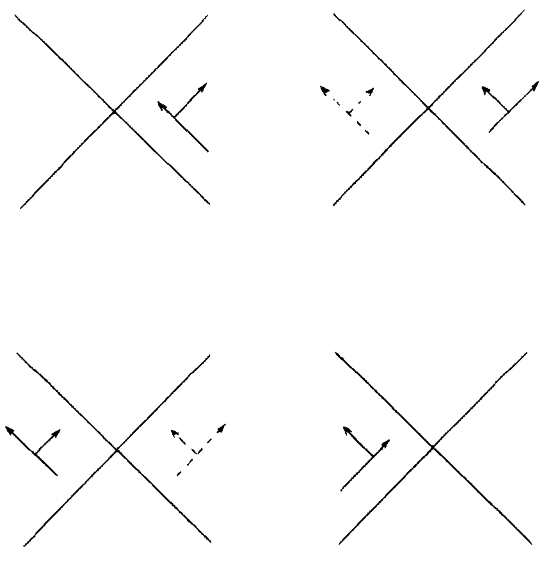

Figure 5. Structure of the basis modes defining the λ vacuum.

simple analyticity property. Nevertheless, it seems physically clear that the resulting state is 'vacuous' on the W-shaped Cauchy surface consisting of the past null infinities and past horizons of L and R. (One should not say 'devoid of particles', because, as is well known for the case of flat space–time (Newton and Wigner 1949, Wightman and Schweber 1955), the relation between particles and fields is non-local; it is unlikely that the concept of a particle at a given point—if it makes sense at all in curved space—is independent of the behaviour of the positive-frequency normal-mode functions in other regions. We have in mind, rather, a concept of vacuity based on the behaviour of local functions of the field, such as the energy–momentum tensor. Here, the point is that conditions in R are identical for the ξ vacuum and the λ vacuum, as far as the field is concerned.) The λ definition is discussed (but not named) in Unruh (1976).

3.4. *The general case—two horizons*

In the situation of § 2.3, where the range of ρ ends at the right in another horizon, all the eigenfrequencies of equation (3.18) are of the double type. The analytic extension of the manifold R, if we choose to identify the L-regions beyond the two horizons, has the structure indicated in figure 2.

As in the case of the generalized Schwarzschild metrics discussed in § 3.3, there is a variety of natural notions of 'positive frequency', each leading to a state which might be distinguished as 'the vacuum'. The η vacuum is based on solutions in R of positive frequency with respect to the Rindler coordinates, and solutions in L of negative frequency. The υ vacuum is based on solutions of positive frequency with respect to the left horizon's Kruskal coordinates; such solutions are analytic in the lower half U_1 plane on the surface $V_1=0$ and analytic in the lower half V_1 plane on the surface $U_1=0$. The ξ vacuum is supposed to be based on functions which are positive-frequency functions on both past horizons of R; these must be analytic in the lower half U_2 plane on $V_2=0$, as well as in the lower half U_1 plane on $V_1=0$. (The right-hand horizon has its own proper set of Kruskal coordinates (V_2, U_2)—in contrast to the null infinity of the Schwarzschild model, where the natural coordinates were the Rindler (v, u), because they asymptotically become Cartesian there.)

We consider now the υ functions, which are positive-frequency functions everywhere near the left horizon, and ask how they behave near the right horizon. These functions are defined in equations (3.20) and (3.22). In R they are (see equation (3.21*a*))

$$
\begin{aligned}
{}^{-}\hat{\phi}_\omega \equiv \hat{\phi}_\omega &= [2\sinh(\pi|\omega|)]^{-1/2}\, e^{\pi\omega/2}\, e^{-i\omega\tau_1} B^{-1}[\vec{\chi}\,\psi_j]^{(*)}, \\
{}^{+}\hat{\phi}_\omega &= [2\sinh(\pi|\omega|)]^{-1/2}\, e^{\pi\omega/2}\, e^{-i\omega\tau_1} B^{-1}[\bar{\chi}\,\psi_j]^{(*)},
\end{aligned}
\tag{3.23}
$$

where '(*)' indicates that the bracketed functions are to be complex-conjugated if ω is negative, but not if ω is positive. Near the right horizon ($\rho_1\to+\infty$) we have, in analogy with equations (3.19)

$$
\vec{\chi}\sim T'\, e^{i|\omega|\rho_1}, \qquad \bar{\chi}\sim e^{-i|\omega|\rho_1}+R'\, e^{i|\omega|\rho_1}. \tag{3.24}
$$

When these are substituted into equations (3.23), one effect of '(*)' is to remove the absolute-value signs from ω. Therefore, as the right past horizon ($V_2=0,\ U_2<0$) is approached, we get

$$
{}^{+}\hat{\phi}_\omega \sim e^{-i\omega\tau_1}\, e^{-i\omega\rho_1} = D^{-i\omega}(-U_2)^{ic\omega} \tag{3.25a}
$$

and ${}^{-}\hat{\phi}_\omega=0$, where equation (2.42) or equation (2.43) has been used. Similarly, near the right future horizon ($U_2=0,\ V_2>0$) we find

$$
{}^{\pm}\hat{\phi}_\omega \sim e^{-i\omega\tau_1}\, e^{i\omega\rho_1} = D^{i\omega} V_2^{-ic\omega}. \tag{3.25b}
$$

The numerical coefficients in ${}^{\nu}\hat{\phi}_\gamma$ and ${}^{\nu}\hat{\phi}_{-\gamma}$ still differ by the factor $e^{\pi\gamma}$.

On the other hand, equations (3.25) show that if ${}^{\nu}\hat{\phi}_\gamma$ and ${}^{\nu}\hat{\phi}_{-\gamma}$ had been constructed by analytic continuation through the right-hand horizon instead of the left, then their magnitudes in region R would have differed by a factor $e^{\pi c\gamma}$. If $c \neq 1$, therefore, the conditions of positive frequency at the two horizons are inconsistent. Vacuum conditions can be imposed on any two adjacent sides of R, but not on all four simultaneously (see figure 2). Thus one has the ξ vacuum, the time-reversed ξ vacuum, the υ vacuum, and the space reflection of the υ vacuum (based on functions of positive frequency around the right horizon). The last of these will be denoted by $\bar{\upsilon}$. The situation is not very different from that for the Schwarzschild metric. The ξ vacuum will exhibit a radiation process of the Hawking type, with fluxes of matter crossing both future horizons (see § 4 for a more precise description). The υ vacuum represents the right horizon as surrounded by a stationary cloud of matter, which may be regarded as composed of coherent radiation going inward and outward.

The 'Planckian' form of the square of the coefficient $[2\sinh(\pi\gamma)]^{-1/2}\, e^{-\pi\gamma/2} = (e^{2\pi\gamma}-1)^{-1/2}$ in the Bogolubov transformations (3.20) and (3.22) leads to the 'thermodynamic' interpretation of the Hawking process. Applying that language to the present case of two horizons with $c \neq 1$, one says that non-trivial effects occur because the two horizons have different temperatures and hence cannot be in equilibrium. The ratio of the temperature of the left horizon to that of the right horizon is

$$T_1/T_2 = c. \tag{3.26}$$

(See equation (2.41), and recall that the temperature of an horizon is inversely proportional to its parameter M or r (Hawking 1975, Gibbons and Hawking 1977).) The absolute magnitudes of the temperatures do not appear in our formulae because our notational conventions (see § 2.2) make τ and ρ, and hence ω and γ, effectively dimensionless numbers. Recall from § 2.3 that c depends not only on the local geometry of the two horizons but also on how they are fitted together globally. A black hole of mass M inside a large, massive galaxy has a lower temperature than a black hole of identical mass far outside the galaxy.

If $c = 1$, then one sees that function (3.21*b*) is the proper continuation of function (3.21*a*) into L through either the left or the right horizon. (That is, the phases, as well as the weight factor $e^{-\pi\gamma}$, match up, so that single-valued $\hat{\phi}$ functions are well defined.) In this case, therefore, the ξ vacuum, the υ vacuum, and their space and time reflections are all the same state. The two black holes are in equilibrium.

We are not compelled, however, to identify the second sheets reached through the two horizons. The analytically extended manifold might be a covering space of the space considered up to now, having sheets . . . , L_1, R_1, L_2, R_2, L_3, . . . , with, possibly, $R_N \equiv R_1$ for some N. Consider such a many-sheeted space–time with $c = 1$. For each γ and ν there are as many positive-norm functions in a basis as there are sheets ($2N$). To define a ξ-υ-vacuum one would require the basis functions to be analytic (in the by now familiar sense) on all horizons. From the previous discussion, however, it is clear that all solutions with this property will be periodic, taking the same form in R_1, R_2, ... There are only two such functions for each γ and ν, analogous to ${}^{\nu}\hat{\phi}_{\gamma}$ and ${}^{\nu}\hat{\phi}_{-\gamma}$ in the two-sheeted case. So it is impossible to choose a complete basis of solutions consisting of functions with the desired analytic property and their complex conjugates. Strangely, *there is no ξ vacuum* for such a model. (This may be related to a previous observation (Davies and Fulling 1977a) on the difficulty of finding vacuum states, invariant under the de Sitter group, for the covering spaces of two-dimensional de Sitter space.)

However, a construction like that of the λ vacuum in § 2.3 is possible here. One can reproduce 'vacuum conditions' in the interior of any sheet (say R_1) by choosing a positive-norm basis which involves the functions ${}^{\nu}_{+}\phi_{\gamma}$ and $({}^{\nu}_{+}\phi_{\gamma})^*$ there always in the ratio $e^{\pi\gamma}$, as in equations (3.20) and (3.22). A definite global extension is not needed in order to discuss the possible quantum states for the algebra of field observables inside R_1.

4. Expectation values of the stress tensor of a two-dimensional massless field

4.1. Conformal vacuum states

The physical significance of the various quantum states defined in § 3 can be investigated easily in the special case of massless scalar fields in two-dimensional space–times. The field equations of these models can be solved explicitly by conformal mappings onto flat space–time: in a conformal null coordinate system, in which the metric takes the form

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu = C\, d\bar{v}\, d\bar{u}, \tag{4.1}$$

the normal modes are just plane waves, proportional to $e^{-i\omega\bar{v}}$ or $e^{-i\omega\bar{u}}$. As a consequence, the solutions involve no scattering; any (classical) radiation present simply propagates either to the left or to the right along null rays. The quantum theories are not trivial, however, since the curvature of space–time may cause creation or annihilation of radiation. We shall adopt the point of view of the references cited at the end of § 1, that most of the relevant physics is summarized in the 'renormalized' energy–momentum tensor, which can be calculated with the aid of a covariant procedure of regularization and subtraction of state-independent divergent terms. In Davies and Fulling (1977a) it is shown that every general solution of the two-dimensional massless field equation by separation of variables in a coordinate system of the type (4.1) gives rise, through equation (3.1), to a vacuum state in which the vacuum expectation value of the stress tensor is

$$\langle T_{\mu\nu}\rangle = \theta_{\mu\nu} - (48\pi)^{-1} R g_{\mu\nu}, \tag{4.2}$$

where

$$\begin{aligned}
\theta_{\bar{v}\bar{v}} &= -(12\pi)^{-1} C^{1/2} \partial_{\bar{v}}^2 (C^{-1/2}),\\
\theta_{\bar{u}\bar{u}} &= -(12\pi)^{-1} C^{1/2} \partial_{\bar{u}}^2 (C^{-1/2}), \qquad (4.3)\\
\theta_{\bar{v}\bar{u}} &= 0,
\end{aligned}$$

$$R = 4C^{-3}(C\partial_{\bar{v}}\partial_{\bar{u}} C - \partial_{\bar{v}} C \partial_{\bar{u}} C). \tag{4.4}$$

The form (4.3) applies when there are no spatial boundary conditions making the spectrum of normal modes discrete.

The tensor (4.2) satisfies

$$\nabla_\mu T^\mu{}_\nu = 0, \qquad T_\mu{}^\mu = R/24\pi; \tag{4.5}$$

the trace is the same for all these conformal vacuum states, since R is just the curvature scalar of the manifold. Therefore, the difference, $\Delta_{\mu\nu}$, between the expectation values of $T_{\mu\nu}$ in two states is conserved and traceless, hence satisfies

$$\partial_u \Delta_{vv} = 0, \qquad \partial_v \Delta_{uu} = 0 \tag{4.6a}$$

in any conformal null coordinate system (see Davies and Fulling 1977a, equations (2.36)). The expectation value of $T_{\mu\nu}$ for a general state in the Fock space associated with a conformal vacuum differs from the $\langle T_{\mu\nu}\rangle$ for that vacuum by the formal expectation value of the correspondingly normal ordered quantity, $:T_{\mu\nu}:$, in that state, which, when finite, is conserved and traceless. Consequently, equations (4.5) and (4.6) hold for all states for which the expectation value of $T_{\mu\nu}$ has been defined. The stress tensors of two states differ only by the contribution of a certain distribution of conserved, massless radiation:

$$\Delta_{vv} = \Delta_{vv}(v), \qquad \Delta_{uu} = \Delta_{uu}(u). \tag{4.6b}$$

Knowing $\langle T_{\mu\nu}\rangle$ for any one state, therefore, one can easily obtain it for another state from sufficient initial data.

In the specific situation treated in this paper, the η vacuum corresponds to choosing $(\bar{v}, \bar{u})$ to be the Rindler coordinates, (v, u), for which $\tau = \frac{1}{2}(v+u)$ is the coordinate conjugate to the Killing vector. Since C is then a function only of $\rho = \frac{1}{2}(v-u)$, one has

$$\theta^{\eta}_{vv} = \theta^{\eta}_{uu} = -(48\pi)^{-1}C^{1/2}\partial^2_\rho(C^{-1/2}) = -(192\pi)^{-1}C^{-2}[3(\partial_\rho C)^2 - 2C\partial^2_\rho C]. \tag{4.7}$$

Referring to equation (2.23), we write (for quadrant R)

$$C = \mathrm{e}^{2\rho}H(-\mathrm{e}^{2\rho}), \qquad H = F^2, \tag{4.8}$$

and obtain

$$\theta^{\eta}_{vv} = \theta^{\eta}_{uu} = -\frac{1}{48\pi}\left[1 + 3\,\mathrm{e}^{4\rho}\left(\frac{H'}{H}\right)^2 - 2\,\mathrm{e}^{4\rho}\frac{H''}{H}\right], \tag{4.9}$$

where the argument of H is $-\mathrm{e}^{2\rho} = VU$.

On the other hand, if $(\bar{v}, \bar{u})$ are the Kruskal coordinates, (V, U), then the positive-frequency normal modes satisfy the analyticity conditions which are used to characterize the υ vacuum. (Indeed, the initial motivation of those conditions was that they are satisfied by the Kruskal plane waves of positive frequency in ultrastatic models.) Therefore, the stress tensor of the υ vacuum is given by equations (4.2) and (4.3) with $C = H(VU)$:

$$\begin{aligned} \theta^{\upsilon}_{VV} &= -\frac{1}{48\pi}U^2\left[3\left(\frac{H'}{H}\right)^2 - 2\frac{H''}{H}\right], \\ \theta^{\upsilon}_{UU} &= -\frac{1}{48\pi}V^2\left[3\left(\frac{H'}{H}\right)^2 - 2\frac{H''}{H}\right]. \end{aligned} \tag{4.10}$$

Comparing with equation (4.9) with the aid of the tensor transformation formulae

$$T_{VV} = V^{-2}T_{vv}, \qquad T_{UU} = U^{-2}T_{uu}, \tag{4.11}$$

we find that

$$\Delta^{\upsilon\eta}_{vv} \equiv \theta^{\upsilon}_{vv} - \theta^{\eta}_{vv} = (48\pi)^{-1} = \Delta^{\upsilon\eta}_{uu}. \tag{4.12}$$

The constancy of $\Delta^{\upsilon\eta}_{\mu\nu}$ in Rindler coordinates is a consequence of equations (4.6b) and the fact that $\theta_{\tau\tau}$ and $\theta_{\tau\rho}$ must be independent of τ for both the η vacuum and the υ vacuum. (Both states are invariant under translations in τ, which are Lorentz boosts

$$V \to \alpha V, \qquad U \to U/\alpha$$

in Kruskal coordinates.)

The application of the general formulae (4.3) to the Kruskal coordinate system is strictly correct only if the Kruskal plane waves are indeed normal modes, satisfying the field equation globally. One must check that there is no trouble caused by a boundary condition at the far right edge of the region R (and the left edge of the postulated symmetric extension, L). In most of the concrete examples mentioned in § 3 all is well, because ρ, and hence $e^{2\rho} = x^2 - t^2$, range all the way to $+\infty$, so that the range of x on a Cauchy surface is $-\infty < x < \infty$ and the initial data can be Fourier transformed in the usual way to express any solution in terms of plane waves. (The possibility of a singularity at some finite ρ in regions P and F is not relevant.) In the case of 'a black hole in a box', however, ρ terminates in R at some finite value where a boundary condition is imposed. In (t, x) space this boundary becomes a 'moving mirror' with trajectory $x^2 - t^2 = \text{constant}$, so that the field equation cannot be solved by separation of variables in the Kruskal system. Nevertheless, in § 4.2 it will become clear that equations (4.10) are valid even in systems of this nature. This fact is a generalization of the observation in Fulling and Davies (1976) that in flat space a uniformly accelerated mirror does not radiate.

4.2. Summation of boost-invariant normal modes

Equations (4.3) do not apply to the ξ vacuum, since that state cannot be defined in terms of plane-wave normal modes in any one conformal coordinate system. Instead, we shall calculate $\langle T_{\mu\nu}\rangle^{\xi}$ by evaluating the appropriate sum over the normal modes $\hat{\phi}_\gamma$ and ${}^{+}_{\pm}\phi_{\pm\omega}$ (see § 3.3 and figure 3). The similar calculation using the modes of figure 4 gives an alternative, and more general, derivation of $\langle T_{\mu\nu}\rangle^{\upsilon}$. We confine attention to the region R, for brevity.

Following precisely the steps in Davies and Fulling (1977a) (which should be consulted for details), we calculate the ξ-vacuum expectation value of the point-split stress tensor,

$$\langle T_{\mu\nu}(x)\rangle_\epsilon = \sum_i T(\phi_i, \phi_i^*)_\epsilon, \tag{4.13}$$

with $T_{vu}(\phi_i, \phi_i^*)_\epsilon = 0$,

$$T_{uu}(\phi_i, \phi_i^*)_\epsilon = \tfrac{1}{2}U_\epsilon U_{-\epsilon}(\partial_u\phi_i(x_\epsilon)\partial_u\phi_i^*(x_{-\epsilon}) + \partial_u\phi_i^*(x_\epsilon)\partial_u\phi_i(x_{-\epsilon})), \tag{4.14}$$

and a similar equation for T_{vv}, where i runs over the positive-norm half of a complete orthonormal set of normal modes, $x_{\pm\epsilon}$ are points displaced a distance ϵ from $x = (v, u)$ along a geodesic, and $U_{\pm\epsilon}$ and $V_{\pm\epsilon}$ are factors introduced by the parallel transport of tensor indices. The only difference from Davies and Fulling (1977a) is in the set of normal modes considered. Let us list the normal modes for the ξ vacuum of an asymptotically flat model. First, we have equation (3.20), specialized to the conformally solvable case and to quadrant R:

$$\hat{\phi}_\omega = [2\sinh(\pi|\omega|)]^{-1/2} e^{\pi\omega/2}(4\pi|\omega|)^{-1/2} e^{-i\omega u} \qquad (-\infty < \omega < \infty); \tag{4.15}$$

these contribute only to the sum for $\langle T_{uu}\rangle_\epsilon$. Second, we have

$${}^{+}_{+}\phi_\omega = (4\pi\omega)^{-1/2} e^{-i\omega v} \qquad (\omega > 0), \tag{4.16}$$

which contributes only to T_{vv}. Finally, there is ${}^{+}_{-}\phi_{-\omega}$, the counterpart of ${}^{+}_{+}\phi_\omega$ concentrated in L, which contributes to neither sum since we consider only x in R. We find

$$\begin{aligned} T_{uu}(\hat{\phi}_\omega, \hat{\phi}_\omega^*) &= [8\pi|\omega|\sinh(\pi|\omega|)]^{-1} e^{\pi\omega}\omega^2 U_\epsilon U_{-\epsilon} \operatorname{Re}(e^{-i\omega\Delta u}), \\ T_{vv}({}^{+}_{+}\phi_\omega, {}^{+}_{+}\phi_\omega^*) &= (4\pi\omega)^{-1}\omega^2 V_\epsilon V_{-\epsilon} \operatorname{Re}(e^{-i\omega\Delta v}) \end{aligned} \tag{4.17}$$

($\Delta u = u_\epsilon - u_{-\epsilon} = O(\epsilon)$, etc), and hence

$$\langle T_{uu}\rangle_\epsilon = \frac{1}{4\pi} U_\epsilon U_{-\epsilon} \operatorname{Re}\left(\int_0^\infty d\omega\, \omega \coth(\pi\omega)\, e^{i\omega\Delta u}\right), \tag{4.18}$$

$$\langle T_{vv}\rangle_\epsilon = \frac{1}{4\pi} V_\epsilon V_{-\epsilon} \operatorname{Re}\left(\int_0^\infty d\omega\, \omega\, e^{i\omega\Delta v}\right) = -\frac{1}{4\pi} V_\epsilon V_{-\epsilon} (\Delta v)^{-2}. \tag{4.19}$$

Thus $\langle T_{vv}\rangle_\epsilon$ is the same as the result of Davies and Fulling (1977a), which applies to the η vacuum. From Gradshteyn and Ryzhik (1965, equation (3.551.3) and § 9.5) we obtain

$$\begin{aligned}\langle T_{uu}\rangle_\epsilon &= (4\pi^3)^{-1} U_\epsilon U_{-\epsilon} \Gamma(2)(\tfrac{1}{2}\zeta(2, \tfrac{1}{2}\beta) - \beta^{-2}) \qquad (\beta \equiv -i\Delta u/\pi) \\ &= -(4\pi)^{-1} U_\epsilon U_{-\epsilon} (\Delta u)^{-2} + (48\pi)^{-1} U_\epsilon U_{-\epsilon} + O(\Delta u).\end{aligned} \tag{4.20}$$

The first term in the final line of equation (4.20) is just the $\langle T_{uu}\rangle_\epsilon$ of the η vacuum. Since $U_\epsilon U_{-\epsilon} = 1 + O(\Delta u)$, all the divergences as $\epsilon \to 0$ are concentrated in that term. Therefore, if we define the limit $\epsilon \to 0$ by the same subtraction *ansatz* used for the η vacuum (see Davies and Fulling 1977a), the conclusion of the calculation is

$$\begin{aligned}&\langle T_{\mu\nu}\rangle^\xi = \langle T_{\mu\nu}\rangle^\eta + \Delta^{\xi\eta}_{\mu\nu}, \\ &\Delta^{\xi\eta}_{vv} = 0, \qquad \Delta^{\xi\eta}_{uu} = (48\pi)^{-1} = \Delta^{\upsilon\eta}_{uu}, \qquad \Delta^{\xi\eta}_{vu} = 0.\end{aligned} \tag{4.21}$$

Clearly, the analogous calculation for the υ vacuum reproduces equations (4.12). That calculation applies regardless of whether the boundary conditions permit the field equation to be solved in terms of Kruskal plane waves.

The result (4.21), or $\theta^\xi_{uu} = \theta^\upsilon_{uu}$, $\theta^\xi_{vv} = \theta^\eta_{vv}$, applies to asymptotically flat cases, such as the Schwarzschild metric. For double black hole models the ξ vacuum is defined differently, and the analogous argument shows that $\theta^\xi_{vv} = \theta^{\bar\upsilon}_{vv}$. (The $\bar\upsilon$ vacuum is the one constructed by imposing the positive-frequency condition at the right-hand horizon.)

4.3. *Minkowski space*

We shall examine the physical significance of the foregoing general results in particular cases, starting with flat space. Interpretation is easier when one considers the components of the energy-momentum tensor relative to an orthonormal frame at each point. To display time-translation symmetry of any of our models, one chooses the time axis of the orthonormal tetrad to point along the Killing vector, ∂_τ. That is, one examines the quantities

$$\hat\theta_{\tau\tau} \equiv e^{-2\rho} H^{-1}(T_{vv} + T_{uu}) = \hat\theta_{\rho\rho}, \qquad \hat\theta_{\tau\rho} \equiv e^{-2\rho} H^{-1}(T_{vv} - T_{uu}). \tag{4.22}$$

These expressions can be misleading, however, at points near the horizon. Since the Killing vector becomes null there, tensor components in that frame suffer 'infinite redshifts and blueshifts', so that a quantity which is physically finite on the horizon may appear to approach infinity or zero there, and vice versa (Fulling 1977). Therefore, one uses instead a frame aligned with the Kruskal coordinate axes:

$$\hat\theta_{tt} \equiv H^{-1}(T_{VV} + T_{UU}) = \hat\theta_{xx}, \qquad \hat\theta_{tx} \equiv H^{-1}(T_{VV} - T_{UU}). \tag{4.23}$$

But since H is normalized to unity on the horizon, in practice a discussion of the null components usually suffices. A non-zero T_{UU} can be interpreted as a rightward flux of energy (***negative*** energy if $T_{UU} < 0$). Similarly, T_{VV} represents a flux to the left.

To obtain the full physical stress tensor one should add to equations (4.22) and (4.23) the trace term

$$-(48\pi)^{-1}R\hat{g}_{\mu\nu}, \tag{4.24}$$

where

$$\hat{g}_{00} = -\hat{g}_{11} = 1, \qquad \hat{g}_{01} = 0 \tag{4.25}$$

in any orthonormal frame. Since this component of the expectation value is the same for all states, it can be ignored in the following discussions.

In two-dimensional Minkowski space, the υ vacuum is the conventional Poincaré-invariant vacuum. Since $H \equiv 1$, our formulae give $\langle T_{\mu\nu}\rangle^{\upsilon} = 0$, as expected. For the η vacuum we find

$$\hat{\theta}^{\eta}_{\tau\tau} = -(24\pi)^{-1}\,\mathrm{e}^{-2\rho} = -(24\pi z^2)^{-1}, \qquad \hat{\theta}^{\eta}_{\tau\rho} = 0, \tag{4.26}$$

as anticipated from related work (Fulling and Davies 1976, Candelas and Raine 1976). As in equation (2.13), z is the proper distance from the focus of the Rindler coordinates (incorrectly described above equation (6.10) of Fulling and Davies (1976) as the distance from the mirror). Except for sign, the energy–momentum at each point is that of a gas of massless particles in its rest frame. The alternative representation,

$$\theta^{\eta}_{VV} = -(48\pi V^2)^{-1}, \qquad \theta^{\eta}_{UU} = -(48\pi U^2)^{-1}, \tag{4.27}$$

decomposes this into fluxes of negative energy running to the left and to the right, the strength of each beam becoming infinite as one approaches the part of the horizon parallel to it.

An alternative interpretation, perhaps not incompatible with a well defined $T_{\mu\nu}$ operator, is that the η definition of positive frequency should be used to analyse local experiments conducted by an observer whose worldline is one of the Killing orbits (z = constant)—i.e., an observer with the uniform acceleration z^{-1}. Various arguments can be given (Unruh 1976, Israel 1976, Davies 1975) that this observer should detect particles with a Planck spectrum of η-energies corresponding to a temperature of $kT = (2\pi z)^{-1}$, if the state of the field is the υ vacuum. The *difference* between the stresses of the υ vacuum and the η vacuum,

$$\hat{\Delta}^{\upsilon\eta}_{\tau\tau} = +(24\pi z^2)^{-1}, \qquad \hat{\Delta}^{\upsilon\eta}_{\tau\rho} = 0, \tag{4.28}$$

can be regarded as the energy–momentum associated with this effective thermal bath, since in two dimensions the energy density of massless radiation in equilibrium at kT is (cf De Witt 1975, equation (137))

$$\frac{1}{\pi}\int_0^{\infty} \frac{\omega\,\mathrm{d}\omega}{\mathrm{e}^{\omega/kT}-1} = \frac{\pi}{6}(kT)^2. \tag{4.29}$$

In Minkowski space the affine parameter on $\mathscr{I}^-$ (past null infinity) is (a multiple of) V, not v. Therefore, the ξ vacuum, defined in the geometrical spirit of Unruh's definition for the Schwarzschild metric, is just the ordinary υ vacuum. The state constructed from left-moving υ modes and right-moving η modes has no apparent physical significance in this case.

4.4. *The two-dimensional Schwarzschild metric*

The expressions given in Davies *et al* (1976) for the vacuum expectation value of $T_{\mu\nu}$

outside a static star also provide $\langle T_{\mu\nu}\rangle^{\eta}$ for the region R of the full Schwarzschild–Kruskal manifold. (The covariant tensor components in Davies *et al* (1976) must be multiplied by $16M^2$ to match the normalization of coordinates in the present paper—see equations (2.26).) We have

$$\hat{\theta}^{\eta}_{\tau\tau}=\frac{1}{24\pi M^2}\left(1-\frac{2M}{r}\right)^{-1}\left[3\left(\frac{M}{r}\right)^4-2\left(\frac{M}{r}\right)^3\right], \qquad \hat{\theta}^{\eta}_{\tau\rho}=0. \tag{4.30}$$

The noteworthy features of $\hat{\theta}^{\eta}_{\tau\tau}$ are that it is negative and approaches zero like r^{-3} as $r\to\infty$ (cf equations (4.26)).

One can obtain $\langle T_{\mu\nu}\rangle^{\upsilon}$ and $\langle T_{\mu\nu}\rangle^{\xi}$ with the help of the difference formulae (4.12) and (4.21). One has

$$\begin{aligned}\hat{\theta}^{\xi}_{\tau\tau}&=\frac{1}{24\pi M^2}\left(1-\frac{2M}{r}\right)^{-1}\left[3\left(\frac{M}{r}\right)^4-2\left(\frac{M}{r}\right)^3+\frac{1}{32}\right]\to\frac{1}{768\pi M^2},\\ \hat{\theta}^{\xi}_{\tau\rho}&=-\frac{1}{24\pi M^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{1}{32}\right)\to-\frac{1}{768\pi M^2},\end{aligned} \tag{4.31}$$

where the limit as $r\to\infty$ is shown. The expressions (4.31) are completely equivalent to those found in Davies *et al* (1976) and Davies (1976) for the late-time limit of $\langle T_{\mu\nu}\rangle$ outside two-dimensional collapsing bodies. This confirms Unruh's (1976) claim that, given a full Schwarzschild solution with past horizon, the ξ-type initial conditions most closely reproduce the situation expected in the case of a collapsing star (Hawking 1975). Similarly, one finds

$$\begin{aligned}\hat{\theta}^{\upsilon}_{\tau\tau}&=\frac{1}{24\pi M^2}\left(1-\frac{2M}{r}\right)^{-1}\left[3\left(\frac{M}{r}\right)^4-2\left(\frac{M}{r}\right)^3+\frac{1}{16}\right]\to\frac{1}{384\pi M^2},\\ \hat{\theta}^{\upsilon}_{\tau\rho}&=0;\end{aligned} \tag{4.32}$$

these formulae also apply to a black hole inside a reflecting spherical box, and, at least approximately for small $r-2M$, to the case of two black holes in equilibrium in the sense of § 2.4. The υ vacuum is the state which corresponds most closely to equilibrium of the black hole with its surroundings. (When the reaction of the scalar field on the metric is taken into account, the equilibrium may be unstable.) The limit values in equations (4.31) and (4.32) correspond, via equation (4.29), to black-body emission from the black hole and to equilibrium thermal radiation, respectively, with $kT=(8\pi M)^{-1}$.

The discussion in Fulling (1977) of the behaviour of the radiation from a collapsing body near the horizon applies also to $\langle T_{\mu\nu}\rangle^{\xi}$, except that now there is a real flux singularity on the past horizon. Near $r=2M$ we have

$$\begin{aligned}\theta^{\xi}_{UU}&=\theta^{\upsilon}_{UU}\to(2048\pi M^4)^{-1}V^{+2},\\ \theta^{\xi}_{VV}&=\theta^{\eta}_{VV}\sim-(48\pi)^{-1}V^{-2}.\end{aligned} \tag{4.33}$$

By construction of the ξ vacuum, there is no flux into R across the past horizon (i.e., $\theta^{\xi}_{UU}=0$ on $V=0$). However, there is a flux parallel to the past horizon which becomes infinite on the horizon itself. This singularity is forced by the geometry of the space–time and the other defining boundary condition, namely that $\theta^{\xi}_{vv}=0$ on $\mathscr{I}^-$.

On the other hand, $\langle T_{\mu\nu}\rangle^{\upsilon}$ is bounded everywhere in R. We have

$$\begin{aligned}\theta^{\upsilon}_{VV}&=(2048\pi M^4)^{-1}U^2 \qquad \text{when } V=0,\\ \theta^{\upsilon}_{UU}&=(2048\pi M^4)^{-1}V^2 \qquad \text{when } U=0.\end{aligned} \tag{4.34}$$

Thus $\theta^{\upsilon}_{\mu\nu}$ is zero at the apex of the horizon, a point which we may take to be truly empty of matter, although the expectation value of the stress tensor there still has the trace component (4.24), with $R=4Mr^{-3}$. Elsewhere, the flux parallel to the nearest part of the horizon is finite but not zero. On the surfaces at infinity, $\mathscr{I}^{\pm}$, $\theta^{\upsilon}_{\mu\nu}$ approaches a finite, constant value, as indicated in equations (4.32).

The various expectation values of $T_{\mu\nu}$ could be calculated in regions L, F, and P by similar methods. However, there is some question, except in the case of the υ vacuum, whether $\langle T_{\mu\nu}\rangle$ is well defined, as a distribution, on test functions whose support intersects the horizon itself, and, if so, whether it has a singularity concentrated there (e.g., $T_{VV}\propto\delta(V)$). The latter seems especially likely for the λ vacuum, which was defined in § 3.3 by fastening together mode functions from the two sheets of the manifold with a rather arbitrary relative phase.

Many of the conclusions of this subsection have been extended to four dimensions (Christensen and Fulling 1977, P Candelas, work in progress).

4.5. de Sitter space

For two-dimensional de Sitter space–time in a Kruskal coordinate system (equation (2.33)) we have $C^{-1/2}=H^{-1/2}=1-\frac{1}{4}VUr^{-2}$, and hence, from equations (4.3),

$$\theta^{\upsilon}_{\mu\nu}=0. \tag{4.35}$$

So $\langle T_{\mu\nu}\rangle^{\upsilon}$ everywhere has the form (4.24)–(4.25), with $R=2r^{-2}$; i.e.,

$$\langle\hat{T}_{\tau\tau}\rangle=-\langle\hat{T}_{\rho\rho}\rangle=-(24\pi r^2)^{-1}, \qquad \langle\hat{T}_{\tau\rho}\rangle=0. \tag{4.36}$$

(Here r is not a coordinate, but a constant, the radius of the de Sitter hyperboloid.) The construction of the υ vacuum is based on a distinguished point ($V=0$, $U=0$), or rather a distinguished antipodal pair of points. Nevertheless, the expectation value of the energy–momentum tensor in this state is invariant under the $SO_0(2,1)$ isometry group of the space. All points and all directions in the universe are equivalent, with respect to $\langle T_{\mu\nu}\rangle^{\upsilon}$ as well as geometrically. Is this quantum state actually invariant under the isometry group in all respects? Is it identical to any of the states found in Davies and Fulling (1977a) by separation of variables in non-static coordinate systems, which yield the same $\langle T_{\mu\nu}\rangle$? Can invariant states of the massless field be constructed by second quantization from irreducible representation spaces of $SO_0(2,1)$, and if so what is their relation to the states associated with conformal coordinate systems? These questions will be left for future investigation. (The answers for mass zero may well be different from those for $m^2>\frac{1}{4}r^{-2}$, corresponding to the principal series of representations.)

For the η vacuum one finds

$$\hat{\theta}^{\eta}_{\tau\tau}=-\hat{\Delta}_{\tau\tau}=-(24\pi r^2)^{-1}\sin^{-2}(z/r), \tag{4.37}$$

where z is the proper distance from either horizon (see equation (2.34) and preceding text). Midway between them, on the line $z=\frac{1}{2}\pi r$, which is a geodesic, one has

$$\hat{\Delta}_{\tau\tau}=\hat{\Delta}_{\rho\rho}=(24\pi r^2)^{-1}, \qquad \hat{\Delta}_{\tau\rho}=0; \tag{4.38}$$

this suggests a temperature of $kT=(2\pi r)^{-1}$ if one regards the η vacuum as somehow fundamental for an observer located at $z=\frac{1}{2}\pi r$. Figari *et al* (1975), studying a massive scalar field in two-dimensional de Sitter space, have shown that the Gibbs (thermal equilibrium) state of that temperature, defined relative to the field Hamiltonian, H_τ, which generates the τ-translation isometry group in region R, is invariant under $SO_0(2,1)$. It is clear by the general argument in Israel (1976) that this state is the υ vacuum of the massive field, restricted to R. (The η vacuum is the ground state of H_τ.) Gibbons and Hawking (1977) have concluded that an observer at $z = \frac{1}{2}\pi r$ will detect thermal radiation with $kT = (2\pi r)^{-1}$; the arguments are the same as those for an accelerated observer in flat space mentioned in § 4.3. In this case the observer's trajectory is geodesic—which makes the conclusion harder to ignore when discussing the observational relevance of our quantum field theory.

The assumption of a fundamental role for H_τ and its eigenstates in the operational interpretation of the theory requires a physical justification. An attempt to provide one is the model particle detector of Unruh (1976) and Gibbons and Hawking (1977). There, however, what is considered is the integrated response over the detector's entire wordline (on which τ is the natural coordinate), despite the fact that elementary-particle events are approximately localized in time as well as space. The special role of H_τ thus appears to be built in from the beginning. Contrast the frequent proposal to extract the physical content of a quantum field theory in curved space by giving a distinguished role to the unit normal vector field over an entire space-like hypersurface ('diagonalizing the instantaneous Hamiltonian'). The philosophy of the present work is that observables pertaining to a space–time point, such as $T_{\mu\nu}(x)$, should be defined covariantly with respect to that point, without reference to a larger submanifold, time-like or space-like, containing that point. Within that framework we have reached the conclusion that the neighbourhood around the focus of a boost-like Killing vector is 'maximally empty' in the corresponding υ vacuum. In two-dimensional de Sitter space this means that the local vacuum of a point x is not the η vacuum associated with any one of the time-like geodesics through x, but rather the υ vacuum based on the boost-like Killing vector which leaves x invariant. Nevertheless, our theory attributes to this state a non-vanishing stress tensor (equation (4.36)), of a form and magnitude not unrelated to (equation (4.38)) what the particle detector is understood to indicate for that same state under the assumption that it is responding to ordinary positive-energy particles. Imagine a detector whose counting rate has been calibrated by exposure to thermal radiation in flat space at various temperatures and relative velocities. Suppose that the detector, now located in de Sitter space, gives the reading associated with the situation (4.38). A single number cannot determine all three components of $T_{\mu\nu}$. (Indeed, one should not expect to measure all components simultaneously, since they do not commute.) It is entirely plausible that the counter reading is equally consistent with the classically forbidden situation (4.36)—perhaps even (though this is not essential to our point of view) that the detector actually measures $\hat{T}_{\rho\rho}$ or $|\hat{T}_{\tau\tau}|$. ('It is impossible to distinguish whether the detector is absorbing particles from a medium or emitting particles into a negative-energy void.') However, the prediction of Gibbons and Hawking (1977) is that the counting rate (with respect to proper time) is independent of the Lorentz frame of the detector. Hence the detector cannot be responding to a classical thermal bath, which has a preferred rest frame; but it might well be responding to a state of affairs characterized by the invariant stress tensor (4.36). Therefore, accepting the cited analysis of particle detectors for the sake of argument, we tentatively find it not incompatible with, and even tending to support, the belief in a unique, observer-independent renormalized energy–momentum tensor, which takes the form (4.36) in the vacuum of two-dimensional de Sitter space.

An example which should be less controversial is the model (2.46), where the central geodesic is surrounded by a flat neighbourhood. Here we find

$$
\begin{aligned}
\theta^{\eta}_{vv} &= \theta^{\eta}_{uu} = -(48\pi)^{-1}\theta(|\hat{z}|-b), \\
\theta^{\upsilon}_{vv} &= \theta^{\upsilon}_{uu} = +(48\pi)^{-1}\theta(b-|\hat{z}|).
\end{aligned}
\tag{4.39}
$$

The υ vacuum stress in the central flat strip is that of an ordinary massless gas with $kT=(2\pi r)^{-1}$, while the η vacuum stress is zero there. The normal modes of the η quantization are ordinary plane waves there, and there is no reason to doubt that the eigenstates of H_τ have their usual significance for microscopic physics inside the flat region. Incidentally, the fact that $\theta^{\upsilon}_{\mu\nu}=0$ outside the central strip is a special feature of the locally de Sitter geometry. In a more general case one would have a finite, non-zero flux parallel to the horizon, as in the υ vacuum of the Schwarzschild model. A similar remark applies to the model studied in the next section.

4.6. Double black holes

Our example of two horizons not in equilibrium is the metric (2.44). Figure 6 represents qualitatively the energy fluxes near the horizons and the signs of the energy densities near the central joint of the model. The explicit expressions will be discussed only briefly.

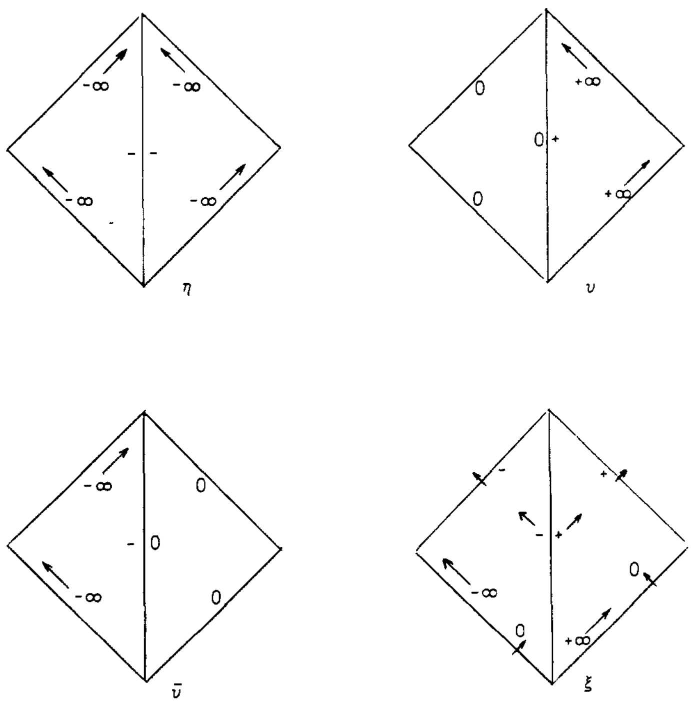

Figure 6. Expected energy densities and fluxes in R in various states of the model (2.44) with $r_2>r_1$. An infinite flux parallel to an horizon is generally accompanied by a finite flux (not shown) across the horizon. In the state ξ, however, there is no flux across the past horizon. The complete emptiness of half of the space in states υ and $\bar{\upsilon}$ is peculiar to the de Sitter geometry.

The η vacuum stress can be calculated, as in any case where the metric is given as a function of the proper distance z (equations (2.21)–(2.22)), from

$$\theta^{\eta}_{vv} = \theta^{\eta}_{uu} = +(96\pi)^{-1} C \partial_z^2(\ln C). \tag{4.40}$$

The result is

$$\theta^{\eta}_{u_1 u_1} = -(48\pi)^{-1}(\theta(-\hat{z}) + c^{-2}\theta(\hat{z})) = c^{-2}\theta^{\eta}_{u_2 u_2}, \tag{4.41}$$

where (§ 2.3) $\hat{z}$ is the distance from the joint, and $c = r_2/r_1$. The stresses of the other states can be found from equation (4.41) from their boundary conditions and the conservation law, as in previous examples. Alternatively, one finds, for example, $\theta^{\bar{\upsilon}}_{\mu\nu}$ in the left half of R by a calculation using equation (2.45) and equation (2.33):

$$\begin{aligned} \theta^{\bar{\upsilon}}_{V_1 V_1} &= -(48\pi)^{-1}(1 - c^{-2}) V_1^{-2} \theta(-\hat{z}), \\ \theta^{\bar{\upsilon}}_{U_1 U_1} &= -(48\pi)^{-1}(1 - c^{-2}) U_1^{-2} \theta(-\hat{z}). \end{aligned} \tag{4.42}$$

When $c = 1$ we recover the de Sitter universe (§ 2.5); the υ and $\bar{\upsilon}$ states are the same. The limit $c \to \infty$ ($r_2 \gg r_1$) is qualitatively like the Schwarzschild model; the right-hand horizon becomes a null infinity, and the η and $\bar{\upsilon}$ states become indistinguishable. In the opposite limit, $r_2 \ll r_1$ (r_1 fixed), one sees that the left-hand black hole is cooler than the other one, and therefore will suffer a *positive* flux of energy over its future horizon if the initial conditions are vacuum:

$$\theta^{\xi}_{V_1 V_1} > 0 \qquad \text{on } U_1 = 0 \qquad (r_2 < r_1). \tag{4.43}$$

In the immediate vicinity of $\hat{z} = 0$ one has

$$\hat{\theta}^{\eta}_{\tau\tau} = -(24\pi)^{-1}(r_1^{-2}\theta(-\hat{z}) + r_2^{-2}\theta(\hat{z})), \tag{4.44}$$

$$\hat{\theta}^{\upsilon}_{\tau\tau} = (24\pi)^{-1}(r_1^{-2} - r_2^{-2})\theta(\hat{z}), \tag{4.45}$$

$$\hat{\theta}^{\bar{\upsilon}}_{\tau\tau} = (24\pi)^{-1}(r_2^{-2} - r_1^{-2})\theta(-\hat{z}), \tag{4.46}$$

$$\theta^{\xi}_{\tau\tau} = (48\pi)^{-1}(r_2^{-2} - r_1^{-2})(\theta(-\hat{z}) - \theta(\hat{z})), \tag{4.47a}$$

$$\hat{\theta}^{\xi}_{\tau\rho} = (48\pi)^{-1}(r_2^{-2} - r_1^{-2}). \tag{4.47b}$$

The significance of equations (4.47) is that if $r_2 > r_1$, the central joint of the space radiates positive energy to the right and negative energy to the left, as indicated in figure 6. One can verify that the full expressions (including factors like $\cos^{-2}(\hat{z}/r_1)$, omitted in equations (4.44)–(4.47)) satisfy the conservation law everywhere, including points on the joint. (Note that the discontinuity in the curvature R on the joint is only a step. The explicit components of $\nabla_\mu T^{\mu}{}_{\nu} = 0$ in null coordinates are given by Davies and Fulling (1977a, equations (2.36)).)

All the states investigated here for $c \neq 1$ must be regarded as singular, in that the expectation value of the stress tensor, physically normalized, becomes infinite near at least two of the four horizon surfaces (see, e.g., equations (4.42)). The same is true of all states which are invariant under translations in τ; the discussion below equation (4.12) shows that the stress of such a state can differ from those studied here only by constant terms in T_{vv} and T_{uu}, and then formulae like equations (4.11) show that such a term which cancels the infinity on one horizon surface will introduce an infinity on the opposite surface (as in the transition from the υ vacuum to the ξ vacuum). However, there are many non-stationary states which are non-singular in this sense. For example, construct a null coordinate system for the entire space–time (cf Davies and Fulling

1977a, appendix) such that the new coördinates are twice-differentiable functions of (V_1, U_1) in a set O_1 excluding the right horizon, and twice differentiable in (V_2, U_2) in O_2 excluding the left horizon, where O_1 and O_2 together cover the space. Then the stress tensor of the corresponding conformal vacuum (from equations (4.1)–(4.4)) is finite, having no singularity worse than the *steps* made inevitable (somewhere) by the discontinuity in the geometry at the joint.

One might well argue that only the non-singular states are physically acceptable states of the field system, for the space–time manifold considered globally. Nevertheless, the singular ∂_τ-invariant states should be useful in discussing physics inside the region R.

5. Physical summary

We consider scalar quantum field theory in a space–time with a prescribed metric. When this background geometry is independent of time (i.e., has a time-like Killing vector, ∂_τ), there is a strong temptation to assimilate the problem to conventional quantum theory, in the usual formulation of which the time evolution is assigned a strikingly fundamental role. In particular, one might expect the group of time translations of the state vectors to be generated by a positive operator (Hamiltonian), whose ground state is the 'vacuum'. Indeed, that is what results (under suitable technical assumptions) from second quantization of the solutions of the field equation which have positive frequency with respect to ∂_τ (i.e., integrals over solutions that depend on the corresponding time coordinate as $e^{-i\omega\tau}$, $\omega > 0$). In other words, one is interpreting the positive-frequency solutions as the possible wavefunctions of particles, as in the ordinary Klein–Gordon theory.

This procedure and interpretation are presumably physically trustworthy when the orbits of the Killing vector are also the *geodesics* normal to each constant-time hypersurface, as in the class of models we have called 'ultrastatic'. In other cases, though, one must question whether τ is really 'time' in such a fundamental sense. Sometimes a Killing vector leaves invariant a 'focus'—a point, or a two-dimensional set which becomes a point in the 'radius'–time plane when two irrelevant 'angular' variables are suppressed. Typically the space–time then divides into a region R where ∂_τ is time-like and future-directed, a region L where it is past-directed, and regions where it is space-like. These are separated by a cone-like ∂_τ-invariant null surface, the horizon, along which ∂_τ is a null vector. The prototype of such a Killing vector is the generator of homogeneous Lorentz transformations in Minkowski space. The locally time-like Killing vectors in the Schwarzschild and de Sitter solutions are other examples of great interest. Obviously, τ has no geometrical right to be called 'time' near a focus; instead, it is a kind of angular polar coordinate. It comes as no surprise, therefore, that the corresponding 'vacuum' state for region R (the η vacuum) is physically peculiar. The Green functions and expectation values of field observables behave near the focus and horizon in singular ways which are not attributable to any peculiarity of the geometry there. At the same time there may be other Killing vectors time-like in regions overlapping R, but with horizons in different places or with no horizons; hence it is clear that the η vacuum has no unique, fundamental significance. De Sitter space provides a particularly good example: all locally time-like Killing vectors are geometrically alike (conjugate under the isometry group); each time-like geodesic is an orbit of one Killing vector (although most orbits are not geodesic); different geodesics through the same point correspond to different horizons, and hence thoroughly different η vacuum states.

This paper investigates how 'vacuum', or 'positive frequency', might be defined near the horizon in a geometrically and physically sensible way, without prejudicial reference to the Killing vector. (The resulting theory will not have a Hamiltonian formulation, in general.) The key observation, due to Unruh (1976), is that the horizon can be embedded in a flat or ultrastatic space–time; equivalently, that each of the two null surfaces making up the horizon is invariant under translation in its affine parameter, V or U. (This is in addition to the original Killing symmetry, which reduces on the horizon to dilation of V accompanied by contraction of U—i.e., a Lorentz transformation in the ultrastatic space.) Initial data on the horizon should be classed as 'positive frequency' (it is proposed) if they correspond to positive-frequency solutions in the ordinary sense in the fictitious ultrastatic model; this amounts to saying that the data on the horizon surface $U = 0$ are positive-frequency functions of V (like e^{-iEV}), and vice versa. The criterion of analyticity in the lower half-plane of V or U enables one to construct a basis of such functions out of the solutions in R and L of definite frequency with respect to ∂_τ.

An interesting question not investigated here is whether a useful *general* notion of *vacuum conditions at a point* can be defined in terms of positive frequency with respect to affine parameters on the light cone of the space–time point considered, even when the cone does not have the symmetries associated with ultrastatic embedding.

In most cases the region R terminates (in the direction away from the horizon) either at another horizon or at infinity. In the latter case the conformal null infinities, $\mathscr{I}^{\pm}$, are in many ways formally analogous to horizon surfaces. (Thus R is to be visualized as a square standing on one corner—cf figures 2 and 6.) If the vacuum state defined as above relative to one horizon coincides with that defined in the analogous way relative to the other horizon, then we say that the two horizons are in equilibrium. (In the terminology of Gibbons and Hawking (1977), they have equal surface gravity.) The physical significance of the equilibrium vacuum state is brought out by explicit calculations of the expectation value of the energy–momentum tensor of a massless scalar field in two-dimensional models. There is no flux of energy across the horizon surfaces, although in general there is a finite flux parallel to (and on) each of the four horizon surfaces, which vanishes as one of the surfaces transverse to the flow is approached. The 'thermal' character of this radiation is visible in the Planckian form of the Bogolubov coefficients describing the relative contributions of η-positive-frequency and η-negative-frequency solutions in R to the basis modes which define the vacuum. Therefore, one expects the essential features found for the two-dimensional massless field to be true of the general case. Since the vacuum stress is well behaved everywhere, this state is physically more acceptable than the original η vacuum. Nevertheless, it probably still does not have a unique, absolute significance, since the defining boundary conditions arbitrarily assign a special role to the horizons. A possible exception to this last remark is two-dimensional de Sitter space, where there are no fluxes anywhere; that model deserves further study.

If the horizons are not in equilibrium, it is not possible to impose the vacuum boundary condition simultaneously on two null surfaces on opposite sides of R. The condition thus suggests four distinct candidates for vacuum state, depending on which adjacent pair of the four horizon surfaces are privileged to be cleansed of their transverse flux. Across the other surfaces a flux of the Hawking type (Hawking 1975) then appears. The states in which the full left or right horizon is privileged at the expense of the other horizon have been called the υ and $\bar{\upsilon}$ vacuum. If the past halves of both horizons are made vacuous, then we have the initial or ξ vacuum. The ξ vacuum is not invariant under time reversal (though it is under τ translation), and it consequently has a non-vanishing energy flux in the rest frame of the Killing vector ($T_{\tau\rho} \neq 0$). The two-dimensional massless calculations show, as expected, that the flux across the privileged horizon surfaces vanishes. In general, however, the fluxes parallel to these surfaces and across the other two can be as singular as in the η vacuum. A 'smooth' quantum state for horizons not in equilibrium is necessarily *not* invariant under the isometry group generated by the Killing vector. In an asymptotically flat model, the coordinates associated with the Killing vector are also the geometrically natural (affine) coordinates on the 'right-hand horizon', $\mathscr{I}^{\pm}$, so the $\bar{\upsilon}$ vacuum is the same as the η vacuum. In this special case the υ vacuum may be considered physically non-singular, since the infinity otherwise encountered at the right horizon is spread out here into a constant Hawking flux, both incoming and outgoing. (In the Killing frame this appears as a static thermal bath at spatial infinity.) For massive fields, boundary conditions on $\mathscr{I}^{\pm}$ must be re-phrased into conditions on the asymptotic behaviour of solutions in real space and time, but then the ξ vacuum and $\bar{\upsilon}$ vacuum are constructed in the same way as before.

We have appealed to a general two-dimensional massless scalar theory, whose principal physical conclusion is that the expectation value of the energy–momentum tensor in any state is a sum of three terms: (i) A pure trace, $T_{uv} \propto Rg_{uv}$, is completely determined by the geometry at the point in question and hence unambiguously can be regarded as 'vacuum polarization'. (ii) A traceless term (with components θ_{uu} and θ_{vv} in null coordinates), which satisfies *together with the trace term* the conservation law $\nabla_\mu T^\mu{}_\nu = 0$, is interpreted as the energy–momentum tensor of the matter created or annihilated by the curvature of space–time. (The trace effectively acts as the source of this radiation.) (iii) An additional traceless term is conserved by itself, and hence has components of the form $T_{uu}(u)$ and $T_{vv}(v)$. It describes massless matter propagating unmolested against the curved space–time background; its value is different in different quantum states. (The classical language of 'matter' or 'radiation' must not be taken too literally, since we are dealing with quantum states which are coherent superpositions of the various classical possibilities.) One could define the vacuum as the state in which term (iii) is zero. In general, *the division of the traceless part into terms (ii) and (iii) is quite arbitrary*, and it is impossible to settle on a particular state as 'the physical vacuum'. In the context of the present work the ξ vacuum can be characterized as the state in which (inside region R, at least) the matter represented by term (2) is entirely created, rather than destroyed.

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